

DECAY ESTIMATES FOR THE SCHRÖDINGER EVOLUTION ON ASYMPTOTICALLY CONIC SURFACES OF REVOLUTION I

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1. INTRODUCTION

It is well-known that the free Schrödinger evolution satisfies the dispersive bound

$$(1.1) \quad \|e^{it\Delta} f\|_{\infty} \lesssim |t|^{-\frac{d}{2}} \|f\|_1$$

where Δ denotes the Laplacean in \mathbb{R}^d . Another instance of such decay bounds are the global Strichartz estimates

$$(1.2) \quad \|e^{it\Delta} f\|_{L^{2+\frac{4}{d}}(\mathbb{R}^{d+1})} \lesssim \|f\|_{L^2(\mathbb{R}^d)}$$

and mixed-norm versions thereof. In this paper we establish a decay estimate (valid for all t), similar to (1.1), for the Schrödinger evolution on a non-compact 2-dimensional manifold with a trapped geodesic. As we shall explain below, the case of the manifold considered in this investigation, as well as the method of proving the decay estimate, are different from the studies in the existing literature concerning Schrödinger evolution on manifolds.

There has been much activity lately around establishing dispersive and Strichartz estimates for more general operators, namely for Schrödinger operators of the form $H = -\Delta + V$ with a decaying potential V or even more general perturbations. The seminal paper here is Jorneé-Soffer-Sogge[13], and we refer the reader to the survey [17] for more recent references in this area.

Around the same time as [13], Bourgain[3] found Strichartz estimates on the torus. This is remarkable, as compact manifolds do not exhibit dispersion as in (1.1) which was always considered a key ingredient for (1.2). The theme of Strichartz estimates on manifolds (both local and global in time) was then developed further in several important papers, see Smith-Sogge[18], Staffilani-Tataru[19], Burq-Gerard-Tzvetkov[4], [5], Hassel-Tao-Wunsch[11], [12], Robbiano-Zuily[14], and Tataru[20]. Gerard[9] reviews some of the recent work in this field.

A recurring theme in this area is the importance of closed geodesics for Strichartz estimates. In fact, it is well-known that the presence of closed geodesics necessarily leads to a loss of derivatives in the Strichartz bounds. The intuition here is that initial data that are highly localized around a closed geodesic and possess high momentum traveling around this geodesic will lead to so-called meta-stable states in the Schrödinger evolution. These are states that remain "coherent" for a long time, which amounts to absence of dispersion during that time, see for example [9]

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(in the classical approximation, dispersive estimates are governed by the Newtonian scattering trajectories - classically speaking, closed geodesics are non-scattering states).

For this reason, many authors have imposed explicit non-trapping conditions, see [18], [11], [12], [15]. The relevance of this condition lies with the construction of a parametrix, which always involves solving for suitable bi-characteristics. On manifolds these bi-characteristics are governed by the geodesics flow in the co-tangent bundle - hence the relevance of closed geodesics.

There is a large body of work on the so-called Kato smoothing estimates where this non-trapping condition also features prominently, see for example Craig-Kappeler-Strauss[6], Doi[8], and Rodnianski-Tao[15].

Thus, the literature on the trapping case is very limited. In addition, we are not aware of a reference that studies (1.1) rather than (1.2) on manifolds, which then necessarily need to be noncompact.

In this paper, we consider surfaces of revolution

$$\mathcal{S} = \{(x, r(x) \cos \theta, r(x) \sin \theta) : -\infty < x < \infty, 0 \leq \theta \leq 2\pi\}$$

with the metric $ds^2 = r^2(x)d\theta^2 + (1+r'(x)^2)dx^2$. Examples of such surfaces abound, and there is no point to developing a theory for all of them simultaneously (consider the cases: $r(x)$ constant, $r(x)$ periodic, and $r(x)$ rapidly growing). Rather, we single out a class of surfaces of revolution which have rather explicit behavior at both ends. Our basic examples are $r(x) = \langle x \rangle^\alpha$ with $0 \leq \alpha < \infty$. In fact, in order not to obscure our ideas by technical details, we will present the main result of this paper for the case $\alpha = 1$. We remark, however, that the method of this paper equally well applies to other values of α .

We now define the class of asymptotically conic manifolds we shall mainly work with.

Definition 1.1. We assume that $\inf_x r(x) > 0$, and asymptotically that

$$(1.3) \quad r(x) = |x|h(x) \quad \text{for } |x| \geq 1$$

where $h(x) = 1 + O(x^{-2})$ and also $h^{(k)}(x) = O(x^{-2-k})$ for all $k \geq 1$. Examples of such $r(x)$ are $r(x) = \sqrt{1 + |x|^2} =: \langle x \rangle$ and variants thereof. Note that our surface \mathcal{S} is asymptotic to cones at both ends. For convenience, we shall also make the symmetry assumption $r(x) = r(-x)$. We shall refer to such a surface of revolution as *asymptotically conic and symmetric*.

The main result of this paper is the following theorem, where $\Delta_{\mathcal{S}}$ denotes the Laplace-Beltrami operator on \mathcal{S} .

Theorem 1.2. *Let \mathcal{S} be a surface of revolution in \mathbb{R}^3 which is asymptotically conic and symmetric as in Definition 1.1. Then for all t*

$$(1.4) \quad \|e^{it\Delta_{\mathcal{S}}} f\|_{L^\infty(\mathcal{S})} \lesssim |t|^{-1} \|f\|_{L^1(\mathcal{S})}$$

provided f does not depend on the angular variable θ .

The symmetry assumption can be easily removed, but we include it to simplify the exposition. Section 2 will be devoted to the proof of this theorem. We remark that for surfaces which are asymptotic to $\langle x \rangle^\alpha$. It is clear that in case $\alpha = 0$ the surface of revolution \mathcal{S} is just the cylinder $S^1 \times \mathbb{R}$, with the metric $ds^2 = d\theta^2 + dx^2$, for which the dispersive estimate with radial

data is the same as the one for the free 1-dimensional Schrödinger operator, namely one has $\|e^{it\Delta_{(S^1 \times \mathbb{R})}} f\|_{L^\infty(S^1 \times \mathbb{R})} \lesssim |t|^{-\frac{1}{2}} \|f\|_{L^1(\mathbb{R})}$. For $0 < \alpha < 1$, the methods of this paper yield the decay rate $t^{-\frac{1}{2}(1+\alpha)}$, whereas for $1 \leq \alpha < \infty$ it is t^{-1} . The intuition concerning these decay rates is as follows: Let $B(p, t)$ denote the volume of a geodesic ball of radius t centered at the point $p \in \mathcal{S}$. Then for fixed p one has

$$\text{vol}(B(p, t)) \sim t^{1+\alpha} \quad \text{or} \quad t^2$$

depending on whether $\alpha < 1$ or $\alpha \geq 1$. In view of the unitarity of the Schrödinger flow, we see that the decay rate should be given by $\text{vol}(B(p, t))^{-\frac{1}{2}}$ and this is indeed the case.

In a sequel to this paper we will discuss the case of non-radial data $f = f(\theta, r)$. In fact, the methods of this paper allow us to prove the following result.

Theorem 1.3. *Let \mathcal{S} be a surface of revolution in \mathbb{R}^3 which is asymptotically conic and symmetric as in Definition 1.1. Then for each integer n there exists a constant $C(n)$ so that for all t*

$$\|e^{it\Delta_{\mathcal{S}}}(e^{in\theta} f)\|_{L^\infty(\mathcal{S})} \leq C(n) |t|^{-1} \|f\|_{L^1(\mathcal{S})}$$

provided f does not depend on the angular variable θ .

Summing in n we of course obtain a global $L^1(\mathcal{S}) \rightarrow L^\infty(\mathcal{S})$ decay estimate with a loss of derivatives in θ . The details, as well as the dependence of the constant $C(n)$ on n will be discussed in the sequel to this paper. Obviously, the behavior of $C(n)$ for large n is very important as it governs how many derivatives (in θ) we will lose on the right-hand side. Note that the loss of derivatives in θ is in agreement with the aforementioned intuition that meta-stable states can form from data with high momentum around a closed geodesic. We do not lose any derivatives in Theorem 1.2 since radial data cannot get trapped in the classical picture (the scattering trajectories are generators of our surface of revolution). Finally, we remark that Theorem 1.2 leads to a Strichartz estimate for radial data using standard techniques.

We now briefly describe the main ideas behind Theorems 1.2 and 1.3. First, using arc-length coordinates ξ on \mathcal{S} and after multiplying by the weight $r^{\frac{1}{2}}(\xi)$, we reduce matters to the Schrödinger operator (with n as in Theorem 1.3)

$$\mathcal{H}_n := -\partial_\xi^2 + V(\xi) + \frac{n^2}{r^2(\xi)}$$

on the line. Here $V(\xi)$ is a smooth potential that behaves like $-\frac{1}{4\xi^2}$ as $\xi \rightarrow \pm\infty$. It is crucial to notice that the combined potential behaves like $\frac{2n^2-1/4}{\xi^2}$ as $\xi \rightarrow \pm\infty$. In order to prove our theorems, we express the resolvent kernel as

$$(\mathcal{H}_n - (\lambda^2 + i0))^{-1}(\xi, \xi') = \frac{f_+(\xi, \lambda)f_-(\xi', \lambda)}{W(\lambda)}$$

when $\xi > \xi'$. Here f_\pm are the usual Jost solutions for \mathcal{H}_n at energy λ^2 which are asymptotic to $e^{\pm i\xi\lambda}$ as $\xi \rightarrow \pm\infty$, and $W(\lambda)$ is the Wronskian of $f_+(\cdot, \lambda)$ with $f_-(\cdot, \lambda)$. It is a well-known fact of scattering theory, see Deift-Trubowitz [7], that for potentials $V(\xi)$ satisfying $\langle \xi \rangle V(\xi) \in L^1(\mathbb{R})$ the Jost solutions exist and are continuous in $\lambda \in \mathbb{R}$; in fact, they are continuous in $\lambda \neq 0$ under the weaker condition $V \in L^1$. Here, this continuity property – as well as the existence statement – fail at $\lambda = 0$ because of the inverse square behavior. Furthermore, it is also common knowledge

that for decay estimates as in Theorem 1.2 and 1.3, the energy $\lambda = 0$ plays a decisive role. For this reason we need to develop some machinery to determine the asymptotic behavior of both $f_{\pm}(\cdot, \lambda)$ as well as $W(\lambda)$ as $\lambda \rightarrow 0$. We accomplish this by means of two types of perturbative arguments. The first type is perturbative in the energy λ and around the zero energy solutions; the latter of course correspond to the harmonic functions on \mathcal{S} of which there are really two for each n : if $n = 0$ the first one is constant and the second is logarithmic – more precisely, it behaves like $\log \xi$ as $\xi \rightarrow \infty$ and like $-\log |\xi|$ as $\xi \rightarrow -\infty$. Starting from these two, we build a fundamental system of solutions to $\mathcal{H}_0 f = \lambda^2 f$ at least in the range $|\xi| \ll \lambda^{-1}$. The second type uses the operator

$$\tilde{\mathcal{H}}_0 := -\partial_{\xi}^2 - \frac{1}{4\xi^2}$$

as approximating operator. The Jost solutions of $\tilde{\mathcal{H}}_0$ are given explicitly in terms of (weighted) Hankel functions of order zero. Using these as approximation, we obtain expressions for the true Jost solutions which are sufficiently accurate in the range $|\xi| \gg \lambda^{-\epsilon}$. It is important that this range overlaps with the range from the previous perturbative argument. Hence, we are able to glue our fundamental systems together to yield global Jost solutions. Finally, proving Theorem 1.2 then reduces to certain oscillatory integrals for which we rely on stationary phase type arguments, see (2.9) below. Note that although these oscillatory integrals are one-dimensional, they still yield the t^{-1} decay due to the fact that they contain weights of the form $(\langle \xi \rangle \langle \xi' \rangle)^{-\frac{1}{2}}$.

2. THE PROOF OF THEOREM 1.2

The Laplace-Beltrami operator on \mathcal{S} is

$$(2.1) \quad \Delta_{\mathcal{S}} = \frac{1}{r(x)\sqrt{1+r'(x)^2}} \partial_x \left(\frac{r(x)}{\sqrt{1+r'(x)^2}} \partial_x \right) + \frac{1}{r^2(x)} \partial_{\theta}^2$$

It is convenient to switch to arclength parametrization. Thus, let

$$\xi(x) = \int_0^x \sqrt{1+r'(y)^2} dy.$$

Then (2.1) can be written as

$$(2.2) \quad \Delta_{\mathcal{S}} = \frac{1}{r(\xi)} \partial_{\xi} (r(\xi) \partial_{\xi}) + \frac{1}{r^2(\xi)} \partial_{\theta}^2$$

where we have abused notation: $r(\xi)$ instead of $r(x(\xi))$. We remark that using (2.2), one obtains two θ independent harmonic functions on \mathcal{S} :

$$(2.3) \quad \begin{aligned} y_0(\xi) &= 1, \\ y_1(\xi) &= \int_0^{\xi} r^{-1}(\xi') d\xi'. \end{aligned}$$

By our asymptotic assumption on $r(x)$,

$$(2.4) \quad \xi(x) = \begin{cases} \sqrt{2}x + c_\infty + O(x^{-1}) & \text{as } x \rightarrow \infty \\ \sqrt{2}x - c_\infty + O(x^{-1}) & \text{as } x \rightarrow -\infty \end{cases}$$

where c_∞ is some constant.

In particular,

$$(2.5) \quad r(\xi) = \frac{1}{\sqrt{2}}\xi \left(1 - \frac{c_\infty}{\xi} + O(\xi^{-2}) \right) \quad \text{as } \xi \rightarrow \infty.$$

Hence,

$$(2.6) \quad y_1(\xi) = \sqrt{2} \operatorname{sign}(\xi)(\log |\xi| + O(1)) \quad \text{as } |\xi| \rightarrow \infty.$$

The appearance of $y_1(\xi)$ is remarkable, as it establishes the existence of a harmonic function on \mathcal{S} , which grows in absolute value like $\log |\xi|$ at both ends. Note that there can be no harmonic function on \mathbb{R}^2 which grows like $\log r$ as $r = |\xi| \rightarrow \infty$; indeed, this would violate the mean value property of harmonic functions (on the other hand, the fundamental solution is $\log r$). In the case of $y_1(\xi)$ it is crucial that it behaves like $-\log |\xi|$ on one end, and like $\log |\xi|$ on the other (note the presence of $\operatorname{sign}(\xi)$) – therefore, y_1 is no contradiction to the mean-value property. One can also think of \mathcal{S} as two planes joined by a neck. Then on the upper plane the harmonic function grows like $\log |\xi|$, whereas on the lower plane it behaves like $-\log |\xi|$.

Setting $\omega(\xi) := \frac{\dot{r}(\xi)}{r(\xi)}$ yields

$$(2.7) \quad \Delta_{\mathcal{S}} y(\xi, \theta) = \partial_\xi^2 y + \omega \partial_\xi y + \frac{1}{r^2} \partial_\theta^2 y.$$

First we need a few simplifications in order to deal with the Laplacean. To this end, we remove the first order term in (2.7) by setting

$$(2.8) \quad y(\xi, \theta) = r(\xi)^{-1/2} u(\xi, \theta).$$

Then

$$(2.9) \quad \partial_\xi^2 y + \omega \partial_\xi y + \frac{1}{r^2} \partial_\theta^2 y = r^{-1/2} [\partial_\xi^2 u - V(\xi)u + \frac{1}{r^2} \partial_\theta^2 u]$$

with

$$(2.10) \quad V(\xi) = \frac{1}{4}\omega^2(\xi) + \frac{1}{2}\dot{\omega}(\xi).$$

Letting $H = -\partial_\xi^2 + V$, and using (2.8) and (2.9), we observe that (1.4) is equivalent with

$$(2.11) \quad \|r^{-1/2} e^{itH} r^{-1/2} u\|_{L^\infty(\mathbb{R})} \lesssim t^{-1} \|u\|_{L^1(\mathbb{R})}.$$

By the usual reduction to the resolvent of H , see Artbazar-Yajima[2], Weder[21], and Goldberg-Schlag[10], the bound (2.11) is equivalent to the following oscillatory integral bound:

$$(2.12) \quad \sup_{\xi > \xi'} \left| \int_0^\infty e^{it\lambda^2} (\langle \xi \rangle \langle \xi' \rangle)^{-\frac{1}{2}} \lambda \operatorname{Im} \left[\frac{f_+(\xi, \lambda) f_-(\xi', \lambda)}{W(\lambda)} \right] d\lambda \right| \\ + \sup_{\xi < \xi'} \left| \int_0^\infty e^{it\lambda^2} (\langle \xi \rangle \langle \xi' \rangle)^{-\frac{1}{2}} \lambda \operatorname{Im} \left[\frac{f_+(\xi', \lambda) f_-(\xi, \lambda)}{W(\lambda)} \right] d\lambda \right| \lesssim t^{-1}$$

where $W(\lambda) := W(f_+(\cdot, \lambda), f_-(\cdot, \lambda))$ is the Wronskian of solutions $f_\pm(\cdot, \lambda)$ of the following ODE

$$(2.13) \quad \begin{aligned} H f_\pm(\xi, \lambda) &= -f_\pm''(\xi, \lambda) + V(\xi) f_\pm(\xi, \lambda) \\ &= \lambda^2 f_\pm(\xi, \lambda) \\ f_\pm(\xi, \lambda) &\sim e^{\pm i\lambda\xi} \quad \text{as } \xi \rightarrow \pm\infty \end{aligned}$$

provided $\lambda \neq 0$. f_\pm are called the *Jost solutions* and it is a standard fact that these solutions exist. Indeed, since $\omega = \frac{\dot{r}}{r}$, in view of (2.10) and (2.5), we see that

$$|V(\xi)| \lesssim \langle \xi \rangle^{-2}.$$

In fact, V decays no faster since

$$(2.14) \quad V(\xi) = -\frac{1}{4\xi^2} + O(\xi^{-3}) \quad \text{as } |\xi| \rightarrow \infty.$$

It is also important to note that the term $O(\xi^{-3})$ behave like a symbol, i.e. $|\frac{d^k}{d\xi^k} O(\xi^{-3})| \lesssim \langle \xi \rangle^{-3-k}$. Therefore, $f_\pm(\cdot, \lambda)$ are solutions of the *Volterra integral equations*

$$(2.15) \quad f_+(\xi, \lambda) = e^{i\lambda\xi} + \int_\xi^\infty \frac{\sin(\lambda(\eta - \xi))}{\lambda} V(\eta) f_+(\eta, \lambda) d\eta$$

and similarly for f_- .

However, these integral equations have no meaning at $\lambda = 0$. In fact, the zero energy solutions of $Hu = 0$ are given by

$$(2.16) \quad \begin{aligned} u_0(\xi) &= r^{1/2}(\xi), \\ u_1(\xi) &= r^{1/2}(\xi) \int_0^\xi r^{-1}(\eta) d\eta, \end{aligned}$$

see (2.3) and (2.8). Since these functions are not asymptotically constant as $|\xi| \rightarrow +\infty$, it follows that (2.13) and (2.15) have no meaning at $\lambda = 0$.

In passing, we remark that due to the special form of V , the Schrödinger operator H can be factorized as

$$(2.17) \quad \begin{aligned} H &= \mathcal{L}^* \mathcal{L}, \\ \mathcal{L} &= \frac{d}{d\xi} - \frac{1}{2}\omega. \end{aligned}$$

In particular, H has no negative spectrum. One can of course recover u_0, u_1 in (2.16) by means of (2.17): first, solve $\mathcal{L}u_0 = 0$ and then observe that $\mathcal{L}^*(\frac{1}{u_0}) = 0$. Therefore, $Hu_0 = 0$ and solving $\mathcal{L}u_1 = \frac{1}{u_0}$ yields $Hu_1 = 0$.

Lemma 2.1. *For any $\lambda \in \mathbb{R}$, define*

$$(2.18) \quad u_j(\xi, \lambda) := u_j(\xi) + \lambda^2 \int_0^\xi [u_1(\xi)u_0(\eta) - u_1(\eta)u_0(\xi)]u_j(\eta, \lambda) d\eta$$

where $j = 0, 1$. Then $Hu_j(\cdot, \lambda) = \lambda^2 u_j(\cdot, \lambda)$ with $u_j(\cdot, 0) = u_j(\cdot)$, for $j = 0, 1$ and

$$(2.19) \quad W(u_0(\cdot, \lambda), u_1(\cdot, \lambda)) = 1$$

for all λ .

Proof. First, one checks that $W(u_0(\cdot), u_1(\cdot)) = 1$. This yields $Hu_j(\cdot, \lambda) = \lambda^2 u_j(\cdot, \lambda)$ since $Hu_j(\cdot) = 0$ for $j = 0, 1$. Second, $u_j(0, \lambda) = u_j(0)$ and $u'_j(0, \lambda) = u'_j(0)$ for $j = 0, 1$. Hence $W(u_0(\cdot, \lambda), u_1(\cdot, \lambda)) = u'_1(0)u_0(0) - u_1(0)u'_0(0) = 1$. \square

As an immediate corollary we have the following statement.

Corollary 2.2. *With $f_\pm(\cdot, \lambda)$ as in (2.13), one has for any $\lambda \neq 0$*

$$(2.20) \quad \begin{aligned} f_+(\xi, \lambda) &= a_+(\lambda)u_0(\xi, \lambda) + b_+(\lambda)u_1(\xi, \lambda) \\ f_-(\xi, \lambda) &= a_-(\lambda)u_0(\xi, \lambda) + b_-(\lambda)u_1(\xi, \lambda) \end{aligned}$$

where $a_\pm(\lambda) = W(f_\pm(\cdot, \lambda), u_1(\cdot, \lambda))$ and $b_\pm(\lambda) = -W(f_\pm(\cdot, \lambda), u_0(\cdot, \lambda))$. Moreover, $a_-(\lambda) = a_+(\lambda)$ and $b_-(\lambda) = -b_+(\lambda)$.

Proof. The Wronskian relations for a_\pm, b_\pm follow immediately from (2.19). Recall that we are assuming $r(x) = r(-x)$ and therefore also $r(\xi) = r(-\xi)$. In particular, this implies that $f_-(-\xi, \lambda) = f_+(\xi, \lambda)$ and $u_0(-\xi) = u_0(\xi)$ as well as $u_1(-\xi) = -u_1(\xi)$. Thus,

$$a_-(\lambda) = W(f_-(\cdot, \lambda), u_1(\cdot, \lambda)) = -W(f_-(-\cdot, \lambda), u_1(-\cdot, \lambda)) = W(f_+(\cdot, \lambda), u_1(\cdot, \lambda)) = a_+(\lambda)$$

and

$$b_-(\lambda) = -W(f_-(\cdot, \lambda), u_0(\cdot, \lambda)) = W(f_-(-\cdot, \lambda), u_0(-\cdot, \lambda)) = W(f_+(\cdot, \lambda), u_0(\cdot, \lambda)) = -b_+(\lambda)$$

as claimed. \square

The point of this corollary is as follows: in order to prove (2.12) we need to obtain detailed understanding of the functions $f_\pm(\xi, \lambda)$. For large ξ , we will obtain asymptotic estimates by perturbing off the potential $-\frac{1}{4\xi^2}$. But this analysis fails for small ξ , so to tackle that problem we can use (2.18) to derive useful bounds. In the end, we need to “glue” these two regions together. This is the meaning of Corollary 2.2.

To achieve our goal, namely to show (2.12), we need a careful analysis of $f_{\pm}(\xi, \lambda)$. We start by rewriting (2.14) as

$$(2.21) \quad V(\xi) = -\frac{1}{4\xi^2} + V_1(\xi), \quad |\xi| > 1$$

where $|V_1(\xi)| \lesssim |\xi|^{-3}$ by Definition 1.1. Moreover, $|V_1^{(k)}(\xi)| \lesssim |\xi|^{-3-k}$ for $|\xi| > 1$. Let $H_0 = -\partial_{\xi}^2 - \frac{1}{4\xi^2}$.

Lemma 2.3. *For any $\lambda > 0$ the problem*

$$\begin{aligned} H_0 f_0(\cdot, \lambda) &= \lambda^2 f_0(\cdot, \lambda), \\ f_0(\xi, \lambda) &\sim e^{i\xi\lambda} \end{aligned}$$

as $\xi \rightarrow \infty$ has a unique solution on $\xi > 0$. It is given by

$$(2.22) \quad f_0(\xi, \lambda) = \sqrt{\frac{\pi}{2}} e^{i\pi/4} \sqrt{\xi\lambda} H_0^{(+)}(\xi\lambda).$$

Here $H_0^{(+)}(z) = J_0(z) + iY_0(z)$ is the Hankel function.

Proof. It is well-known, see Abramowitz-Stegun[1], that the ODE

$$w''(z) + \left(\lambda^2 + \frac{1}{4z^2} \right) W(z) = 0$$

has a fundamental system of solutions $\sqrt{z} J_0(\lambda z)$, $\sqrt{z} Y_0(\lambda z)$ or equivalently, $\sqrt{z} H_0^{(+)}(\lambda z)$, $\sqrt{z} H_0^{(-)}(\lambda z)$.

Recall the asymptotics

$$\begin{aligned} H_0^{(+)}(x) &\sim \sqrt{\frac{2}{\pi x}} e^{i(x-\frac{\pi}{4})} & \text{as } x \rightarrow +\infty \\ H_0^{(-)}(x) &\sim \sqrt{\frac{2}{\pi x}} e^{-i(x-\frac{\pi}{4})} & \text{as } x \rightarrow +\infty. \end{aligned}$$

Thus, (2.22) is the unique solution so that

$$f_0(\xi, \lambda) \sim e^{i\xi\lambda},$$

as claimed. □

As was observed in (2.15), The Volterra type integral equations play a crucial role in our analysis throughout the paper so for the convenience of the reader we will very briefly sketch how one solves such equations. Let us consider the following Volterra equations

$$(*) \quad f(x) = g(x) + \int_x^\infty K(x, s) f(s) ds,$$

or

$$(**) \quad f(x) = g(x) + \int_a^x K(x, s) f(s) ds,$$

with some $g(x) \in L^\infty$ and $a \in \mathbb{R}$. Evidently, one solves them by an iteration procedure which requires finding a suitable convergent majorant for the resulting series expansion. In Lemma 2.4 below we show that, depending on the choice of norm of K , this majorant is either a geometric series or an exponential series. We remark that – as usual – one only needs the latter alternative in this paper since it does not require a smallness condition.

Lemma 2.4. *Let $a \in \mathbb{R}$ and $g(x) \in L^\infty(a, \infty)$.*

- *If $M := \sup_{x>a} \int_x^\infty |K(x, s)| ds < 1$, then there exists a unique L^∞ solution to the Volterra equation (*), valid for $x > a$, which is given by*

$$f(x) = g(x) + \sum_{n=1}^{\infty} \int_x^\infty K^n(x, s) g(s) ds,$$

with $K^1(x, s) = K(x, s)$ and $K^n(x, s) = \int_x^\infty K^{n-1}(x, t) K(t, s) dt$ for $n \geq 2$. One also has the bound

$$\|f\|_{L^\infty(a, \infty)} \leq (1 - M)^{-1} \|g\|_{L^\infty(a, \infty)},$$

*and a similar statement holds for (**).*

- *Let $\mu := \int_a^\infty \sup_{a < x < s} |K(x, s)| ds < \infty$. Then there exists a unique solution to (*) given by*

$$(2.23) \quad f(x) = g(x) + \sum_{n=1}^{\infty} \int_a^\infty \dots \int_a^\infty \prod_{i=1}^n \chi_{[x_{i-1} < x_i]} K(x_{i-1}, x_i) g(x_n) dx_n \dots dx_1.$$

Furthermore one has the bound

$$\|f\|_{L^\infty(a, \infty)} \leq e^\mu \|g\|_{L^\infty(a, \infty)},$$

*and an analogue statement holds for (**).*

Proof. We only prove the lemma for (*) since the proof for (**) is almost identical. The solutions to both equations are found through a standard Picard-Volterra iteration procedure, which we will from now on refer to as the *Volterra iteration*. For (*), this iteration yields a solution via the Picard-Banach fixed point theorem. We also have

$$(2.24) \quad \left| \int_x^\infty K^n(x, s) g(s) ds \right| \leq M^n \|g\|_{L^\infty},$$

for $x > a$. So if $M < 1$ then the series representing the solution $f(x)$, will converge absolutely and uniformly for $x > a$ and the formula for the sum of a geometric series provides us with the bound for the L^∞ norm of f .

For the second part, we show that the infinite Volterra iteration (2.23) for (*) converges. To this end, define

$$K_0(s) := \sup_{a < x < s} |K(x, s)|$$

Then

$$\begin{aligned}
& \left| \int_a^\infty \cdots \int_a^\infty \prod_{i=1}^n \chi_{[x_{i-1} < x_i]} K(x_{i-1}, x_i) g(x_n) dx_n \dots dx_1 \right| \\
& \leq \int_a^\infty \cdots \int_a^\infty \prod_{i=1}^n \chi_{[x_{i-1} < x_i]} K_0(x_i) |g(x_n)| dx_n \dots dx_1 \\
& = \|g\|_{L^\infty(a, \infty)} \frac{1}{n!} \int_a^\infty \cdots \int_a^\infty \prod_{i=1}^n K_0(x_i) dx_n \dots dx_1 \\
& = \frac{1}{n!} \|g\|_{L^\infty(a, \infty)} \left(\int_a^\infty K_0(s) ds \right)^n
\end{aligned}$$

Hence, the series in (2.23) converges absolutely and uniformly in $x > a$ with the uniform upper bound

$$\|g\|_{L^\infty(a, \infty)} \sum_{n=0}^{\infty} \frac{1}{n!} \mu^n = e^\mu \|g\|_{L^\infty(a, \infty)}$$

as claimed. \square

Having these tools at our disposal, we proceed with our investigation of the Jost solutions. To this end, instead of the Volterra equation (2.15) we will work with the following representation of the solutions of (2.13):

Lemma 2.5. *For any $\xi > 0$, $\lambda > 0$,*

$$(2.25) \quad f_+(\xi, \lambda) = f_0(\xi, \lambda) + \int_\xi^\infty G_0(\xi, \eta; \lambda) V_1(\eta) f_+(\lambda, \eta) d\eta$$

with V_1 as in (2.21), f_0 as in (2.22) and

$$(2.26) \quad G_0(\xi, \eta; \lambda) = [\overline{f_0(\xi, \lambda)} f_0(\lambda, \eta) - f_0(\xi, \lambda) \overline{f_0(\lambda, \eta)}] (-2i\lambda)^{-1}.$$

Proof. Simply observe that G_0 is the Green's function of our problem relative to H_0 . Indeed,

$$\begin{aligned}
G_0(\xi, \xi; \lambda) &= 0, \\
\partial_\xi G_0(\xi, \eta; \lambda)|_{\eta=\xi} &= 1, \\
H_0 G_0(\cdot, \eta; \lambda) &= \lambda^2 G_0(\cdot, \eta; \lambda).
\end{aligned}$$

Here we have used that $W(f_0(\cdot, \lambda), \overline{f_0(\cdot, \lambda)}) = -2i\lambda$ which can be seen by computing the Wronskian at $\xi = \infty$.

In conclusion,

$$H_0 f_+(\xi, \lambda) = \lambda^2 \left[f_0(\xi, \lambda) + \int_\xi^\infty G_0(\xi, \eta; \lambda) V_1(\eta) f_+(\lambda, \eta) d\eta \right] - V_1(\xi) f_+(\xi, \lambda)$$

or equivalently,

$$H f_+(\cdot, \lambda) = \lambda^2 f_+(\cdot, \lambda).$$

Finally, observe that for $\xi > \lambda^{-1}$ fixed,

$$\sup_{\eta > \xi} |G_0(\xi, \eta; \lambda)| \lesssim \lambda^{-1}.$$

By the Volterra iteration discussed above, this implies that $|f_+(\xi, \lambda) - f_0(\xi, \lambda)| \lesssim \lambda^{-1} \xi^{-2}$. In particular,

$$f_+(\xi, \lambda) \sim e^{i\lambda\xi} \quad \text{as } \xi \rightarrow \infty$$

and we are done. \square

For small arguments the Hankel function $H_0(z)$ displays the following asymptotic behavior, see [1]:

$$(2.27) \quad H_0^{(+)}(z) = 1 + O_{\mathbb{R}}(z^2) + \frac{2}{\pi} i \log z + i\kappa + iO_{\mathbb{R}}(z^2 \log z)$$

as $z \rightarrow 0$ where κ is some real constant.

Estimating the oscillatory integrals will require understanding $\partial_\lambda^k \partial_\xi^l f_\pm(\xi, \lambda)$, for $0 \leq k + l \leq 2$, $W(\lambda)$, $W'(\lambda)$ and thus $a_\pm(\lambda)$, $b_\pm(\lambda)$, $a'_\pm(\lambda)$ and $b'_\pm(\lambda)$. To obtain asymptotic expansions for all these functions, we need to know the asymptotic behavior of $u_j(\xi)$, and thereafter that of $\partial_\lambda^k \partial_\xi^l u_j(\xi, \lambda)$, for $j = 1, 2$ and $0 \leq k + l \leq 2$.

We start by analyzing the $u_j(\xi)$'s.

Lemma 2.6.

$$(2.28) \quad r(\xi) = \frac{1}{\sqrt{2}} \xi \left(1 - \frac{c_\infty}{\xi} + O(\xi^{-2}) \right) \quad \text{as } \xi \rightarrow \infty.$$

With c_∞ as in (2.4), In particular,

$$(2.29) \quad \begin{aligned} u_0(\xi) &= 2^{-1/4} \xi^{1/2} \left(1 - \frac{c_\infty}{2\xi} + O(\xi^{-2}) \right) \quad \text{as } \xi \rightarrow \infty \\ u_1(\xi) &= 2^{1/4} \xi^{1/2} \left(1 - \frac{c_\infty}{2\xi} + O(\xi^{-2}) \right) (\log \xi + c_2 + O(\xi^{-1})). \end{aligned}$$

Here c_2 is some constant. Moreover, the O -terms behave like symbols.

Proof. In view of (1.3) and (2.4),

$$\begin{aligned} \xi(x) &= \sqrt{2}x + c_\infty + O(x^{-1}) \\ r(x) &= x + O(x^{-1}) \end{aligned} \quad \text{both as } x \rightarrow \infty$$

where the O -terms behave like symbols. This implies (2.28), as well as the expansion of $u_0(\xi) = \sqrt{r(\xi)}$ in (2.29).

Next compute

$$\begin{aligned} \int_0^\xi r^{-1}(\eta) d\eta &= \int_0^\xi \sqrt{2} \langle \eta \rangle^{-1} \left(1 + \frac{c_\infty}{\langle \eta \rangle} + O(\langle \eta \rangle^{-2}) \right) d\eta \\ &= \sqrt{2} (\log \xi + c_2) + O(\xi^{-1}) \quad \text{as } \xi \rightarrow \infty. \end{aligned}$$

Thus,

$$\begin{aligned} u_1(\xi) &= \sqrt{r(\xi)} \int_0^\xi r^{-1}(\eta) d\eta \\ &= 2^{1/4} \xi^{1/2} \left(1 - \frac{c_\infty}{2\xi} + O(\xi^{-2}) \right) (\log \xi + c_2 + O(\xi^{-1})) \quad \text{as } \xi \rightarrow \infty. \end{aligned}$$

□

To study the behavior of $u_j(\xi, \lambda)$'s we recall the Volterra equation (2.18)

$$u_j(\xi, \lambda) := u_0(\xi) + \lambda^2 \int_0^\xi [u_1(\xi)u_0(\eta) - u_1(\eta)u_0(\xi)]u_j(\eta, \lambda) d\eta.$$

Hence, setting $h_j(\xi, \lambda) := \frac{u_j(\xi, \lambda)}{u_j(\xi)}$, for $\xi > 0$ we obtain the integral equations

$$(2.30) \quad h_0(\xi, \lambda) = 1 + \frac{\lambda^2}{u_0(\xi)} \int_0^\xi [u_1(\xi)u_0^2(\eta) - u_0(\xi)u_1(\eta)u_0(\eta)]h_0(\eta, \lambda) d\eta,$$

$$(2.31) \quad h_1(\xi, \lambda) = 1 + \frac{\lambda^2}{u_1(\xi)} \int_0^\xi [u_1(\xi)u_0(\eta)u_1(\eta) - u_0(\xi)u_1^2(\eta)]h_1(\eta, \lambda) d\eta.$$

Therefore, in order to solve the integral equation for $u_j(\xi, \lambda)$ for large ξ , it is enough to carry out the Volterra iteration for (2.30) and (2.31) which are simpler since the first iterates in both cases are identically equal to 1, and then multiply the h_j 's so obtained by the $u_j(\xi)$'s of (2.29). For the Volterra iterations we would need to understand the behavior of the kernels in the integral equations (2.30) and (2.31). For this purpose, Lemma 2.6 yields

Corollary 2.7. *As $\xi \rightarrow \infty$,*

$$(2.32) \quad u_1(\xi) \int_0^\xi u_0^2(\eta) d\eta - u_0(\xi) \int_0^\xi u_1 u_0(\eta) d\eta = \frac{1}{4} 2^{-1/4} \xi^{5/2} + O(\xi^{3/2} \log \xi)$$

$$(2.33) \quad u_1(\xi) \int_0^\xi u_0 u_1(\eta) d\eta - u_0(\xi) \int_0^\xi u_1^2(\eta) d\eta = \frac{1}{4} 2^{1/4} \xi^{5/2} \log \xi + c_3 \xi^{5/2} + O(\xi^{3/2} \log \xi)$$

Proof. In view of the first equality of (2.29) and a justifiable modification of that expression at 0, we have

$$\begin{aligned} \int_0^\xi u_0^2(\eta) d\eta &= 2^{-1/2} \int_0^\xi \eta \left(1 - \frac{c_\infty}{\langle \eta \rangle} + O(\langle \eta \rangle^{-2}) \right) d\eta \\ &= 2^{-1/2} \left(\frac{1}{2} \xi^2 - c_\infty \cdot \xi + O(\log \xi) \right) \\ \int_0^\xi u_0(\eta) u_1(\eta) d\eta &= \int_0^\xi \eta \left(1 - \frac{c_\infty}{\langle \eta \rangle} + O(\langle \eta \rangle^{-2}) \right) (\log \eta + c_2 + O(\langle \eta \rangle^{-1})) d\eta \\ &= \frac{1}{2} \xi^2 \log \xi + \frac{1}{2} \left(c_2 - \frac{1}{2} \right) \xi^2 + O(\xi \log \xi). \end{aligned}$$

Thus,

$$\begin{aligned}
(2.32) &= 2^{-1/4} \xi^{1/2} (\log \xi + c_2 + O(\xi^{-1} \log \xi)) \left(\frac{1}{2} \xi^2 + O(\xi) \right) \\
&\quad - 2^{-1/4} \xi^{1/2} (1 + O(\xi^{-1})) \left(\frac{1}{2} \xi^2 \log \xi + \frac{1}{2} \left(c_2 - \frac{1}{2} \right) \xi^2 + O(\xi \log \xi) \right) \\
&= 2^{-1/4} \xi^{1/2} \left[\frac{1}{4} \xi^2 + O(\xi \log \xi) \right]
\end{aligned}$$

Next, compute

$$\begin{aligned}
\int_0^\xi u_1^2(\eta) d\eta &= \sqrt{2} \int_0^\xi \eta (\log^2 \eta + 2c_2 \log \eta + O(\langle \eta \rangle^{-1} \log \eta)) (1 + O(\langle \eta \rangle^{-1})) d\eta \\
&= \sqrt{2} \left(\frac{1}{2} \xi^2 \log^2 \xi + (2c_2 - 1) \int_0^\xi \eta \log \eta d\eta + O(\xi \log^2 \xi) \right) \\
&= \sqrt{2} \left(\frac{1}{2} \xi^2 \log^2 \xi + \frac{2c_2 - 1}{2} \xi^2 \log \xi - \frac{2c_2 - 1}{4} \xi^2 + O(\xi \log^2 \xi) \right)
\end{aligned}$$

Thus, (2.33) =

$$\begin{aligned}
&2^{1/4} \xi^{1/2} (\log \xi + c_2 + O(\xi^{-1})) (1 + O(\xi^{-1})) \left(\frac{1}{2} \xi^2 \log \xi + \frac{1}{2} \left(c_2 - \frac{1}{2} \right) \xi^2 + O(\xi \log \xi) \right) \\
&\quad - 2^{1/4} \xi^{1/2} (1 + O(\xi^{-1})) \left(\frac{1}{2} \xi^2 \log^2 \xi + \frac{2c_2 - 1}{2} \xi^2 \log \xi - \frac{2c_2 - 1}{4} \xi^2 + O(\xi \log^2 \xi) \right) \\
&= 2^{1/4} \xi^{1/2} \left\{ \frac{1}{2} \xi^2 \log^2 \xi + \frac{2c_2 - 1}{2} \xi^2 \log \xi + O(\xi \log^2 \xi) + \frac{c_2}{2} \left(c_2 - \frac{1}{2} \right) \xi^2 \right. \\
&\quad \left. - \frac{1}{2} \xi^2 \log^2 \xi - \frac{2c_2 - 1}{2} \xi^2 \log \xi + \frac{2c_2 - 1}{4} \xi^2 \right\} \\
&= 2^{1/4} \sqrt{\xi} \left(\frac{1}{4} \xi^2 \log \xi + 2^{-1/4} c_3 \xi^2 + O(\xi \log \xi) \right)
\end{aligned}$$

as claimed. \square

Thus a Volterra iteration and the preceding yields the following result for the $u_j(\xi, \lambda)$'s. The importance of Corollary 2.8 lies with the fact that we do not lose $\log \xi$ factors in the $O(\cdot)$ -terms as such factors would destroy the dispersive estimate. It is easy to see that carrying out the Volterra iteration crudely, by putting absolute values inside the integrals, leads to such $\log \xi$ losses. Therefore, we actually need to compute the Volterra iterates in (2.23) explicitly (or, more precisely, its analogue for (**)) .

Corollary 2.8. *In the range $1 \ll \xi \ll \lambda^{-1}$, $j = 0, 1$,*

$$\begin{aligned}
(2.34) \quad u_j(\xi, \lambda) &= u_j(\xi) (1 + O((\xi \lambda)^2)) \\
\partial_\xi u_j(\xi, \lambda) &= u'_j(\xi) (1 + O((\xi \lambda)^2))
\end{aligned}$$

$$(2.35) \quad \begin{aligned} \partial_\lambda u_0(\xi, \lambda) &= \frac{1}{2} 2^{-1/4} \lambda (\xi^{5/2} + O(\xi^{3/2} \log \xi)) (1 + O((\xi \lambda)^2)) \\ \partial_\lambda u_1(\xi, \lambda) &= \frac{1}{2} 2^{1/4} \lambda (\xi^{5/2} \log \xi + c_3 \xi^{5/2} + O(\xi^{3/2} \log \xi)) (1 + O((\xi \lambda)^2)) \end{aligned}$$

$$(2.36) \quad \begin{aligned} \partial_{\lambda \xi}^2 u_0(\xi, \lambda) &= \frac{5}{4} 2^{-1/4} \lambda (\xi^{3/2} + O(\xi^{1/2} \log \xi)) (1 + O((\xi \lambda)^2)) \\ \partial_{\lambda \xi}^2 u_1(\xi, \lambda) &= \frac{5}{4} 2^{1/4} \lambda (\xi^{3/2} \log \xi + \frac{2}{5} \xi^{3/2} + c_3 \xi^{3/2} + O(\xi^{1/2} \log \xi)) (1 + O((\xi \lambda)^2)) \end{aligned}$$

Proof. We sketch the proof of this somewhat computational lemma, for the function $u_1(\xi, \lambda)$ since the argument for $u_0(\xi, \lambda)$ is completely analogous and in fact easier. The proof of the first equality in (2.34) is based on the Volterra integral equation (2.31)

$$(2.37) \quad h_1(\xi, \lambda) = 1 + \lambda^2 \int_0^\xi \left[\frac{u_1(\xi) u_0(\eta) u_1(\eta) - u_0(\xi) u_1^2(\eta)}{u_1(\xi)} \right] h_1(\eta, \lambda) d\eta$$

and its derivatives in both ξ and λ and the Volterra iteration, for which we also need to use Corollary 2.7. The iteration will produce a solution which is given by

$$\begin{aligned} h_1(\xi, \lambda) = & 1 + \sum_{n=1}^{\infty} \lambda^{2n} \int_0^\xi \int_0^{\xi_1} \dots \int_0^{\xi_{n-1}} \frac{u_1(\xi) u_0(\xi_1) u_1(\xi_1) - u_0(\xi) u_1^2(\xi_1)}{u_1(\xi)} \dots \\ & \frac{u_1(\xi_{n-1}) u_0(\xi_n) u_1(\xi_n) - u_0(\xi_{n-1}) u_1^2(\xi_n)}{u_1(\xi_{n-1})} d\xi_n \dots d\xi_1 = \\ & 1 + \lambda^2 \int_0^\xi \frac{u_1(\xi) u_0(\xi_1) u_1(\xi_1) - u_0(\xi) u_1^2(\xi_1)}{u_1(\xi)} d\xi_1 + \\ & \lambda^4 \int_0^\xi \int_0^{\xi_1} \frac{u_1(\xi) u_0(\xi_1) u_1(\xi_1) - u_0(\xi) u_1^2(\xi_1)}{u_1(\xi)} \times \\ & \frac{u_1(\xi_1) u_0(\xi_2) u_1(\xi_2) - u_0(\xi_1) u_1^2(\xi_2)}{u_1(\xi_1)} d\xi_2 d\xi_1 + \dots \end{aligned}$$

Therefore, (2.33) and the equalities

$$\begin{aligned} u_0(\xi) &= 2^{-1/4} \xi^{1/2} \left(1 - \frac{c_\infty}{2\xi} + O(\xi^{-2}) \right) \\ u_1(\xi) &= 2^{1/4} \xi^{1/2} \left(1 - \frac{c_\infty}{2\xi} + O(\xi^{-2}) \right) (\log \xi + c_2 + O(\xi^{-1})) \end{aligned}$$

yield

$$\begin{aligned} h_1(\xi, \lambda) = & 1 + \frac{\lambda^2}{u_1(\xi)} \left(\frac{1}{4} 2^{1/4} \xi^{5/2} \log \xi + c_3 \xi^{5/2} + O(\xi^{3/2} \log \xi) \right) + \\ & \lambda^4 \left\{ \int_0^\xi u_0(\xi_1) \left[\frac{1}{4} 2^{1/4} \xi_1^{5/2} \log \xi_1 + c_3 \xi_1^{5/2} + O(\xi_1^{3/2} \log \xi_1) \right] d\xi_1 - \right. \\ & \left. \frac{u_0(\xi)}{u_1(\xi)} \int_0^\xi u_1(\xi_1) \left[\frac{1}{4} 2^{1/4} \xi_1^{5/2} \log \xi_1 + c_3 \xi_1^{5/2} + O(\xi_1^{3/2} \log \xi_1) \right] d\xi_1 \right\} + \dots = 1 + O(\lambda^2 \xi^2), \end{aligned}$$

since we are assuming that $1 \ll \xi \ll \lambda^{-1}$. The point to notice here is that terms involving $\xi^4 \log \xi$ (the leading orders) after the integration cancel. Furthermore, we obtain the usual $n!$ gain from the Volterra iteration, see Lemma 2.4, from repeated integration of powers rather than from symmetry considerations. Hence $u_1(\xi, \lambda) = u_1(\xi)(1 + O(\lambda^2 \xi^2))$ in that range. To deal with the derivatives, it is more convenient to differentiate directly the integral equation (2.18) for $u_1(\xi, \lambda)$ with respect to ξ and/or λ , which yields respectively

$$(2.38) \quad \partial_\xi u_1(\xi, \lambda) = \partial_\xi u_1(\xi) + \lambda^2 \int_0^\xi [\partial_\xi u_1(\xi) u_0(\eta) - u_1(\eta) \partial_\xi u_0(\xi)] u_1(\eta, \lambda) d\eta$$

$$(2.39) \quad \begin{aligned} \partial_\lambda u_1(\xi, \lambda) = & 2\lambda \int_0^\xi [u_1(\xi) u_0(\eta) - u_1(\eta) u_0(\xi)] u_1(\eta, \lambda) d\eta + \\ & \lambda^2 \int_0^\xi [u_1(\xi) u_0(\eta) - u_1(\eta) u_0(\xi)] \partial_\lambda u_1(\eta, \lambda) d\eta, \end{aligned}$$

and

$$(2.40) \quad \begin{aligned} \partial_{\lambda\xi}^2 u_1(\xi, \lambda) = & 2\lambda \int_0^\xi [\partial_\xi u_1(\xi) u_0(\eta) - u_1(\eta) \partial_\xi u_0(\xi)] u_1(\eta, \lambda) d\eta + \\ & \lambda^2 \int_0^\xi [\partial_\xi u_1(\xi) u_0(\eta) - u_1(\eta) \partial_\xi u_0(\xi)] \partial_\lambda u_1(\eta, \lambda) d\eta. \end{aligned}$$

In dealing with (2.38), we simply plug in the information from the first equality of (2.34) and calculate the resulting integral. For (2.39), we observe that by (2.33) the term

$$2\lambda \int_0^\xi [u_1(\xi) u_0(\eta) - u_1(\eta) u_0(\xi)] u_1(\eta, \lambda) d\eta$$

is equal to $\lambda(\frac{1}{2}2^{1/4}\xi^{5/2} \log \xi + 2c_3\xi^{5/2} + O(\xi^{3/2} \log \xi))$. Therefore to solve (2.39), one needs to run the Volterra iteration with this expression as the first iterate. The treatment of (2.40) is similar to that of (2.39) and we skip the details. \square

We now turn to $f_\pm(\xi, \lambda)$ as well as $a_\pm, b_\pm(\lambda)$.

Lemma 2.9. *If $\lambda > 0$ is small, and $1 \ll \xi \ll \lambda^{-1}$, then*

$$f_+(\xi, \lambda) = f_0(\xi, \lambda) + O(\xi^{-1/2} \lambda^{\frac{1}{2}-\varepsilon})$$

with $\varepsilon > 0$ arbitrary.

Proof. We observed above that, c.f. (2.25) and (2.26),

$$|G_0(\xi, \eta; \lambda)| \lesssim \sqrt{\xi\eta} |\log \lambda|^2 \chi_{[\xi < \eta < \lambda^{-1}]} + \sqrt{\frac{\xi}{\lambda}} |\log \lambda| \chi_{[\eta > \lambda^{-1}]}$$

Thus integrating and taking $1 \ll \xi \ll \lambda^{-1}$ into account, we obtain

$$\begin{aligned} \left| \int_{\xi}^{\infty} G_0(\xi, \eta; \lambda) V_1(\eta) f_+(\eta, \lambda) d\eta \right| &\lesssim \int_{\xi}^{\lambda^{-1}} \sqrt{\xi \eta} |\log \lambda|^2 \eta^{-3} \sqrt{\eta \lambda} |\log \lambda|^3 d\eta \\ &\quad + \int_{\lambda^{-1}}^{\infty} \sqrt{\frac{\xi}{\lambda}} |\log \lambda| \eta^{-3} d\eta \\ &\lesssim \xi^{-1/2} \lambda^{\frac{1}{2}-\varepsilon}, \end{aligned}$$

as claimed. \square

Lemma 2.10. *For small $\lambda > 0$,*

$$(2.41) \quad \begin{aligned} a_+(\lambda) &= 2^{1/4} c_0 \sqrt{\lambda} (1 + ic_1 \log \lambda + ic_3) + O(\lambda^{1-\varepsilon}) \\ b_+(\lambda) &= i 2^{-1/4} c_0 c_1 \sqrt{\lambda} + O(\lambda^{1-\varepsilon}), \end{aligned}$$

where $c_0 = \sqrt{\frac{\pi}{2}} e^{i\frac{\pi}{4}}$, $c_1 = \frac{2}{\pi}$, and c_3 is some real constant.

Proof. By Corollary 2.2 we have $a_+(\lambda) = f_+(\xi, \lambda) u'_1(\xi, \lambda) - f'_+(\xi, \lambda) u_1(\xi, \lambda)$. Hence Lemma 2.9 and Corollary 2.8 with $\xi = \lambda^{-1/2}$ yield,

$$\begin{aligned} c_0^{-1} 2^{1/4} a_+ &= \sqrt{\lambda \xi} H_0(\xi \lambda) \frac{1}{2} \xi^{-1/2} (\log \xi + c_2 + 2) \\ &\quad - \left(\frac{1}{2} \xi^{-1/2} \sqrt{\lambda} H_0(\xi \lambda) + \sqrt{\xi \lambda} H'_0(\xi \lambda) \lambda \right) \xi^{1/2} (\log \xi + c_2) + O(\lambda^{1-\varepsilon}) \\ &= \sqrt{\lambda} H_0(\xi \lambda) - \sqrt{\xi \lambda} \frac{ic_1}{\xi} \sqrt{\xi} (\log \xi + c_2) + O(\lambda^{1-\varepsilon}) \\ &= \sqrt{\lambda} (1 + ic_1 \log(\xi \lambda) + i\kappa - ic_1 \log \xi - ic_1 c_2) + O(\lambda^{1-\varepsilon}) \\ &= \sqrt{\lambda} (1 + ic_1 \log \lambda + ic_3) + O(\lambda^{1-\varepsilon}), \end{aligned}$$

as claimed. Note that $c_3 = \kappa - c_1 c_2$.

Similarly,

$$\begin{aligned} -c_0^{-1} 2^{\frac{1}{4}} b_+ &= \sqrt{\lambda \xi} H_0(\xi \lambda) \frac{1}{2} \xi^{-1/2} - \xi^{1/2} \left(\frac{1}{2} \xi^{-1/2} \sqrt{\lambda} H_0(\lambda \xi) + \sqrt{\xi \lambda} H'_0(\xi \lambda) \lambda \right) \\ &\quad + O(\lambda^{1-\varepsilon}) \\ &= -\xi \sqrt{\lambda} \frac{ic_1}{\xi \lambda} \lambda + O(\lambda^{1-\varepsilon}) \\ &= -ic_1 \sqrt{\lambda} + O(\lambda^{1-\varepsilon}), \end{aligned}$$

and the lemma follows. \square

Using the expressions for a_+ and b_+ above, we obtain the following

Corollary 2.11. *Let $\lambda > 0$ be small. Then*

$$(2.42) \quad f_+(\xi, \lambda) = c_0 \sqrt{\lambda \langle \xi \rangle} \left(1 + ic_1 \log(\lambda \langle \xi \rangle) + ic_4 + O(\lambda^{\frac{1}{2}-\varepsilon}) + O(\langle \xi \rangle^{-1} \log \langle \xi \rangle) \right)$$

for $0 < \xi < \lambda^{-1}$, whereas for $-\lambda^{-1} < \xi < 0$,

$$(2.43) \quad f_+(\xi, \lambda) = c_0 \sqrt{\lambda \langle \xi \rangle} \left(1 + ic_1 \log(\lambda \langle \xi \rangle^{-1}) + ic_5 + O(\lambda^{\frac{1}{2}-\varepsilon}) + O(\langle \xi \rangle^{-1} \log \langle \xi \rangle) \right)$$

Proof. This follows by inserting our asymptotic expansions for $a_+(\lambda)$, $b_+(\lambda)$, $u_0(\xi, \lambda)$, and $u_1(\xi, \lambda)$ into (2.20). \square

We also need some information about certain partial derivatives of $f_+(\xi, \lambda)$. This is provided by

Lemma 2.12. *For $\lambda > 0$ small and $1 \ll \xi \ll \lambda^{-1}$ we have*

$$\begin{aligned} \partial_\xi f_+(\xi, \lambda) &= \partial_\xi f_0(\xi, \lambda) + O(\xi^{-3/2} \lambda^{\frac{1}{2}-\varepsilon}) \\ \partial_\lambda f_+(\xi, \lambda) &= \partial_\lambda f_0(\xi, \lambda) + O(\xi^{-1/2} \lambda^{-\frac{1}{2}-\varepsilon}) \\ \partial_{\xi\lambda}^2 f_+(\xi, \lambda) &= \partial_{\xi\lambda}^2 f_0(\xi, \lambda) + O(\xi^{-3/2} \lambda^{-\frac{1}{2}-\varepsilon}) \end{aligned}$$

with $\varepsilon > 0$ arbitrary.

Proof. This follows by taking derivatives in Lemma 2.9. \square

To be able to carry out the analysis, one also needs to understand the derivative of the Wronskian. To that end we have

Corollary 2.13. *For small $\lambda > 0$,*

$$(2.44) \quad \begin{aligned} a'_+(\lambda) &= \frac{1}{2} 2^{1/4} c_0 \lambda^{-1/2} (1 + ic_3 + 2ic_1 + ic_1 \log \lambda) + O(\lambda^{-\varepsilon}) \\ b'_+(\lambda) &= \frac{i}{2} 2^{-1/4} c_0 c_1 \lambda^{-1/2} + O(\lambda^{-\varepsilon}) \end{aligned}$$

where $\varepsilon > 0$ is arbitrary.

Proof. In view of the preceding,

$$(2.45) \quad \begin{aligned} a'_+(\lambda) &= W(\partial_\lambda f_+, u_1) + W(f_+, \partial_\lambda u_1) \\ &= W(\partial_\lambda f_0, u_1) + W(f_0, \partial_\lambda u_1) + O(\lambda^{-\varepsilon}) \\ &= \partial_\lambda [c_0 \sqrt{\lambda \xi} H_0(\lambda \xi)] \left(\frac{1}{2} \xi^{-1/2} (\log \xi + c_2) + \xi^{-1/2} \right) 2^{1/4} \\ &\quad - \partial_{\lambda \xi}^2 [c_0 \sqrt{\lambda \xi} H_0(\lambda \xi)] \xi^{1/2} (\log \xi + c_2) \cdot 2^{1/4} \\ &\quad + c_0 \sqrt{\lambda \xi} H_0(\lambda \xi) \cdot \frac{5}{4} \cdot 2^{1/4} \lambda \left(\xi^{3/2} \log \xi + \left(\frac{2}{5} + c_3 \right) \xi^{3/2} \right) \\ &\quad - c_0 \partial_\xi [\sqrt{\lambda \xi} H_0(\lambda \xi)] \frac{1}{2} 2^{1/4} \lambda (\xi^{5/2} \log \xi + c_3 \xi^{5/2}) + O(\lambda^{-\varepsilon}). \end{aligned}$$

Evaluating at $\xi = \lambda^{-1/2}$, one obtains that the third and fourth terms in (2.45) are $O(\lambda^{\frac{1}{2}-\varepsilon})$, and thus error terms. Thus,

$$\begin{aligned}
2^{-1/4}c_0^{-1}a'_+(\lambda) &= \left(\frac{1}{2}\lambda^{-1/2}(1 + ic_1 \log(\lambda\xi) + i\kappa) + ic_1\lambda^{-1/2}\right) \left(\frac{1}{2}(c_2 + \log \xi) + 1\right) \\
&\quad - \left(\frac{1}{4}\lambda^{-1/2}(1 + ic_1 \log(\lambda\xi) + i\kappa) + ic_1\lambda^{-1/2}\right) (\log \xi + c_2) \\
&\quad + O(\lambda^{-\varepsilon}) \\
&= \frac{1}{2}\lambda^{-1/2}(1 + ic_1 \log(\lambda\xi) + i\kappa) + ic_1\lambda^{-1/2} \\
&\quad - \frac{ic_1}{2}\lambda^{-1/2}(\log \xi + c_2) + O(\lambda^{-\varepsilon}) \\
&= \frac{1}{2}\lambda^{-1/2}(1 + ic_1 \log \lambda + i\kappa + 2ic_1 - ic_1c_2) + O(\lambda^{-\varepsilon}) \\
&= \frac{1}{2}\lambda^{-1/2}(1 + ic_3 + 2ic_1 + ic_1 \log \lambda) + O(\lambda^{-\varepsilon}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
2^{-1/4}c_0^{-1}b'_+(\lambda) &= \left(\frac{1}{2}\lambda^{-1/2}(1 + ic_1 \log(\lambda\xi) + i\kappa) + ic_1\lambda^{-1/2}\right) \left(\frac{1}{2}\right) \\
&\quad - \left(\frac{1}{4}\lambda^{-1/2}(1 + ic_1 \log(\lambda\xi) + i\kappa) + ic_1\lambda^{-1/2}\right) + O(\lambda^{-\varepsilon}) \\
&= -\frac{1}{2}ic_1\lambda^{-1/2} + O(\lambda^{-\varepsilon}),
\end{aligned}$$

as claimed. \square

Having this and an explicit expression for the Wronskian $W(\lambda)$ in terms of a_+ and b_+ , we obtain

Corollary 2.14. *For small λ ,*

$$\begin{aligned}
W(\lambda) &= 2\lambda \left(1 + ic_3 + i\frac{2}{\pi} \log \lambda\right) + O(\lambda^{\frac{3}{2}-\varepsilon}) \\
W'(\lambda) &= 2 \left(1 + ic_3 + i\frac{2}{\pi} + i\frac{2}{\pi} \log \lambda\right) + O(\lambda^{\frac{1}{2}-\varepsilon})
\end{aligned}$$

with $\varepsilon > 0$ arbitrary.

Proof. Follows immediately from

$$W(\lambda) = -2a_+b_+(\lambda)$$

and (2.41), (2.44). \square

To estimate the oscillatory integral (2.12) for $|\xi\lambda| > 1$ we also need the following lemma

Lemma 2.15. *Let $m_+(\xi, \lambda) := e^{-i\lambda\xi}f_+(\xi, \lambda)$. Then, provided $\lambda > 0$ is small and $\lambda\xi > 1$,*

$$\begin{aligned}
(2.46) \quad |m_+(\xi, \lambda) - 1| &\lesssim (\lambda\xi)^{-1} \\
|\partial_\lambda m_+(\xi, \lambda)| &\lesssim \lambda^{-2}\xi^{-1}
\end{aligned}$$

Proof. From (2.25), and with $m_0(\xi, \lambda) = e^{-i\lambda\xi} f_0(\xi, \lambda)$,

$$(2.47) \quad m_+(\xi, \lambda) = m_0(\xi, \lambda) + \int_{\xi}^{\infty} \tilde{G}_0(\xi, \eta; \lambda) V_1(\eta) m_+(\eta, \lambda) d\eta$$

where

$$(2.48) \quad \tilde{G}_0(\xi, \eta; \lambda) = \frac{m_0(\xi, \lambda) \overline{m_0(\eta, \lambda)} - e^{-2i(\xi-\eta)\lambda} \overline{m_0(\xi, \lambda)} m_0(\eta, \lambda)}{-2i\lambda}$$

Now, by asymptotic properties of the Hankel functions,

$$m_0(\xi, \lambda) = 1 + O((\xi\lambda)^{-1})$$

where the O -term behaves like a symbol.

Inserting this bound into (2.48) yields

$$|\tilde{G}_0(\xi, \eta; \lambda)| \lesssim \eta$$

provided $\eta > \xi > \lambda^{-1}$. Thus, from (2.47),

$$|m_+(\xi, \lambda) - m_0(\xi, \lambda)| \lesssim \xi^{-1}$$

and thus, for all $\xi\lambda > 1$,

$$|m_+(\xi, \lambda) - 1| \lesssim (\xi\lambda)^{-1}$$

as claimed.

Next, one checks that for $\eta > \xi > \lambda^{-1}$,

$$|\partial_{\lambda} \tilde{G}_0(\xi, \eta; \lambda)| \lesssim \frac{\eta}{\lambda}.$$

Thus, for all $\lambda\xi > 1$,

$$\begin{aligned} |\partial_{\lambda} m_+(\xi, \lambda)| &\lesssim \lambda^{-2} \xi^{-1} + \int_{\xi}^{\infty} |\partial_{\lambda} \tilde{G}_0(\xi, \eta; \lambda)| \eta^{-3} d\eta \\ &\quad + \int_{\xi}^{\infty} \eta^{-2} |\partial_{\lambda} m_+(\eta, \lambda)| d\eta \\ &\lesssim \lambda^{-2} \xi^{-1} + \lambda^{-1} \xi^{-1} + \int_{\xi}^{\infty} \eta^{-2} |\partial_{\lambda} m_+(\eta, \lambda)| d\eta \\ &\lesssim \lambda^{-1} (\lambda\xi)^{-1}, \end{aligned}$$

as claimed. □

We now commence with proving (2.12) for small energies. Thus, let χ be a smooth cut-off function to small energies, i.e., $\chi(\lambda) = 1$ for small $|\lambda|$ and χ vanishes outside a small interval around zero. In addition, we introduce the cut-off functions $\chi_{[|\xi\lambda| < 1]}$ and $\chi_{[|\xi\lambda| > 1]}$ which form a partition of unity adapted to these intervals. It will suffice to consider the case $\xi > \xi'$ in (2.12).

Lemma 2.16. *For all $t > 0$*

$$(2.49) \quad \sup_{\xi, \xi'} \left| \int_0^{\infty} e^{it\lambda^2} \lambda \chi(\lambda) \chi_{[|\xi\lambda| < 1, |\xi'\lambda| < 1]} ((\xi)\langle\xi'\rangle)^{-1/2} \operatorname{Im} \left[\frac{f_+(\xi, \lambda) f_-(\xi', \lambda)}{W(\lambda)} \right] d\lambda \right| \lesssim t^{-1}$$

Proof. We first observe the following:

$$\begin{aligned}
& \operatorname{Im} \left[\frac{f_+(\xi, \lambda) f_-(\xi', \lambda)}{W(\lambda)} \right] \\
&= \operatorname{Im} \left[\frac{(a_+(\lambda) u_0(\xi, \lambda) + b_+(\lambda) u_1(\xi, \lambda))(a_+(\lambda) u_0(\xi', \lambda) - b_+(\lambda) u_1(\xi', \lambda))}{-2a_+ b_+(\lambda)} \right] \\
&= -\frac{1}{2} \operatorname{Im} \left(\frac{a_+}{b_+}(\lambda) \right) u_0(\xi, \lambda) u_0(\xi', \lambda) + \frac{1}{2} \operatorname{Im} \left(\frac{b_+}{a_+}(\lambda) \right) u_1(\xi, \lambda) u_1(\xi', \lambda).
\end{aligned}$$

Further, by (2.41),

$$\begin{aligned}
-\frac{1}{2} \operatorname{Im} \left(\frac{a_+}{b_+}(\lambda) \right) &= \frac{\pi}{2\sqrt{2}} \operatorname{Re} \left[\frac{1 + ic_1 \log \lambda + ic_3 + O(\lambda^{\frac{1}{2}-\varepsilon})}{1 + O(\lambda^{\frac{1}{2}-\varepsilon})} \right] \\
&= O(\lambda^{\frac{1}{2}-\varepsilon}) + \frac{\pi}{2\sqrt{2}}
\end{aligned}$$

and by Corollary 2.13, the O -term can be formally differentiated, i.e.,

$$\frac{d}{d\lambda} \left\{ -\frac{1}{2} \operatorname{Im} \left(\frac{a_+}{b_+}(\lambda) \right) \right\} = O(\lambda^{-\frac{1}{2}-\varepsilon}).$$

Similarly,

$$\frac{1}{2} \operatorname{Im} \left(\frac{b_+}{a_+}(\lambda) \right) = -\frac{\pi}{2\sqrt{2}} \frac{1}{1 + c_1^2 \log^2 \lambda} + O(\lambda^{\frac{1}{2}-\varepsilon})$$

which can again be formally differentiated.

By the estimates of Corollary 2.8, provided $|\xi\lambda| + |\xi'\lambda| < 1$,

$$|u_0(\xi, \lambda) u_0(\xi', \lambda)| \lesssim \sqrt{\langle \xi \rangle \langle \xi' \rangle}$$

and

$$\begin{aligned}
|\partial_\lambda [u_0(\xi, \lambda) u_0(\xi', \lambda)]| &\lesssim \lambda \left(\langle \xi \rangle^{5/2} \langle \xi' \rangle^{1/2} + \langle \xi' \rangle^{5/2} \langle \xi \rangle^{1/2} \right) \\
&\lesssim \lambda \sqrt{\langle \xi \rangle \langle \xi' \rangle} (\langle \xi \rangle^2 + \langle \xi' \rangle^2).
\end{aligned}$$

Similarly,

$$|u_1(\xi, \lambda) u_1(\xi', \lambda)| \lesssim \sqrt{\langle \xi \rangle \langle \xi' \rangle} (\log \lambda)^2$$

and

$$|\partial_\lambda [u_1(\xi, \lambda) u_1(\xi', \lambda)]| \lesssim \lambda \sqrt{\langle \xi \rangle \langle \xi' \rangle} (\langle \xi \rangle^2 + \langle \xi' \rangle^2) (\log \lambda)^2.$$

Hence, integrating by parts in (2.49) yields

$$\begin{aligned}
(2.49) &\lesssim \\
&t^{-1} \int_0^\infty \left| \partial_\lambda [\chi(\lambda) \chi_{[|\xi\lambda|, |\xi'\lambda| < 1]}] (\langle \xi \rangle \langle \xi' \rangle)^{-1/2} \operatorname{Im} \left(\frac{a_+}{b_+}(\lambda) \right) u_0(\xi, \lambda) u_0(\xi', \lambda) \right| d\lambda \\
&+ t^{-1} \int_0^\infty \left| \partial_\lambda [\chi(\lambda) \chi_{[|\xi\lambda|, |\xi'\lambda| < 1]}] (\langle \xi \rangle \langle \xi' \rangle)^{-1/2} \operatorname{Im} \left(\frac{b_+}{a_+}(\lambda) \right) u_1(\xi, \lambda) u_1(\xi', \lambda) \right| d\lambda \\
&\lesssim 1 + \int_0^\infty \chi(\lambda) \lambda (\langle \xi \rangle^2 + \langle \xi' \rangle^2) \chi_{[|\xi\lambda|, |\xi'\lambda| < 1]} d\lambda \\
&\lesssim 1,
\end{aligned}$$

and the lemma is proved. \square

Next, we consider the case $|\xi\lambda| > 1$ and $|\xi'\lambda| > 1$. With the convention that $f_{\pm}(\xi, -\lambda) = \overline{f_{\pm}(\xi, \lambda)}$ we can remove the imaginary part in (2.12) and integrate λ over the whole axis. We shall follow this convention hence forth. To estimate the oscillatory integrals, we shall repeatedly use the following version of stationary phase, see Lemma 2 in [16] for the proof.

Lemma 2.17. *Let $\phi(0) = \phi'(0) = 0$ and $1 \leq \phi'' \leq C$. Then*

$$(2.50) \quad \left| \int_{-\infty}^{\infty} e^{it\phi(x)} a(x) dx \right| \lesssim \delta^2 \left\{ \int \frac{|a(x)|}{\delta^2 + |x|^2} dx + \int_{|x|>\delta} \frac{|a'(x)|}{|x|} dx \right\}$$

where $\delta = t^{-1/2}$.

Using Lemma 2.17 we can prove the following:

Lemma 2.18. *For all $t > 0$*

$$(2.51) \quad \sup_{\xi > 0 > \xi'} \left| \int_{-\infty}^{\infty} e^{it\lambda^2} \lambda \chi(\lambda) \chi_{[|\xi\lambda|>1, |\xi'\lambda|>1]} (\langle \xi \rangle \langle \xi' \rangle)^{-1/2} \frac{f_+(\xi, \lambda) f_-(\xi', \lambda)}{W(\lambda)} d\lambda \right| \lesssim t^{-1}$$

Proof. Writing $f_+(\xi, \lambda) = e^{i\xi\lambda} m_+(\xi, \lambda)$, $f_-(\xi, \lambda) = e^{-i\xi\lambda} m_-(\xi, \lambda)$ as in Lemma 2.15, we express (2.51) in the form

$$(2.52) \quad \left| \int_{-\infty}^{\infty} e^{it\phi(\lambda)} a(\lambda) d\lambda \right| \lesssim t^{-1}$$

where $\xi > 0 > \xi'$ are fixed, $\phi(\lambda) := \lambda^2 + \frac{\lambda}{t}(\xi - \xi')$, and

$$a(\lambda) = \lambda \chi(\lambda) \chi_{[|\xi\lambda|>1, |\xi'\lambda|>1]} (\langle \xi \rangle \langle \xi' \rangle)^{-1/2} \frac{m_+(\xi, \lambda) m_-(\xi', \lambda)}{W(\lambda)}.$$

Let $\lambda_0 = -\frac{\xi - \xi'}{2t}$. We have the bounds

$$(2.53) \quad |a(\lambda)| \lesssim (\langle \xi \rangle \langle \xi' \rangle)^{-\frac{1}{2}} \chi(\lambda) \chi_{[|\xi\lambda|>1, |\xi'\lambda|>1]}.$$

By Corollary 2.14, for small $|\lambda|$

$$\left| \left(\frac{\lambda}{W(\lambda)} \right)' \right| \lesssim \frac{1}{|\lambda|(\log |\lambda|)^2}$$

and by Lemma 2.15, for $|\xi\lambda| > 1$, $|\xi'\lambda| > 1$,

$$|\partial_{\lambda}[m_+(\xi, \lambda) m_-(\xi', \lambda)]| \lesssim \lambda^{-2}(\xi^{-1} + |\xi'|^{-1}).$$

Hence,

$$(2.54) \quad |a'(\lambda)| \lesssim (\langle \xi \rangle \langle \xi' \rangle)^{-1/2} \chi(\lambda) \chi_{[|\xi\lambda|>1, |\xi'\lambda|>1]} \left\{ \frac{|\lambda|^{-1}}{|\log \lambda|^2} + \lambda^{-2}(\xi^{-1} + |\xi'|^{-1}) \right\}.$$

We will need to consider three cases in order to prove (2.52) via (2.50), depending on where λ_0 falls relative to the support of a .

Case 1: $|\lambda_0| \lesssim 1$, $|\lambda_0| \gtrsim |\xi|^{-1} + |\xi'|^{-1}$.

Note that the second inequality here implies that

$$\frac{\xi + |\xi'|}{t} \gtrsim \frac{\xi + |\xi'|}{\xi |\xi'|}$$

or

$$1 \gtrsim \frac{t}{\xi |\xi'|}.$$

Furthermore, we remark that $a \equiv 0$ unless $\xi \gtrsim 1$ and $|\xi'| \gtrsim 1$.

Starting with the first integral on the right-hand side of (2.50) we conclude from (2.53) that

$$\int \frac{|a(\lambda)|}{|\lambda - \lambda_0|^2 + \delta^2} \lesssim (\langle \xi \rangle \langle \xi' \rangle)^{-1/2} t^{1/2} \lesssim 1.$$

From the second integral we obtain from (2.54) that

$$\begin{aligned} \int_{|\lambda - \lambda_0| > \delta} \frac{|a'(\lambda)|}{|\lambda - \lambda_0|} d\lambda &\lesssim (\langle \xi \rangle \langle \xi' \rangle)^{-1/2} \delta^{-1} \int \frac{\chi(\lambda) d\lambda}{|\lambda| (\log |\lambda|)^2} \\ &\quad + (\langle \xi \rangle \langle \xi' \rangle)^{-1/2} (\langle \xi \rangle^{-1} + \langle \xi' \rangle^{-1}) \delta^{-1} \int_{\lambda > \xi^{-1} + |\xi'|^{-1}} \frac{d\lambda}{\lambda^2} \\ &\lesssim \sqrt{\frac{t}{\langle \xi \rangle \langle \xi' \rangle}} \\ &\lesssim 1. \end{aligned}$$

Case 2: $|\lambda_0| \lesssim 1$, $|\lambda_0| \ll \langle \xi \rangle^{-1} + \langle \xi' \rangle^{-1}$.

Then $|\lambda - \lambda_0| \sim |\lambda|$ on the support of a , which implies that

$$\begin{aligned} \int \frac{|a(\lambda)|}{|\lambda - \lambda_0|^2 + t^{-1}} d\lambda &\lesssim (\langle \xi \rangle \langle \xi' \rangle)^{-1/2} \int_{\lambda > \xi^{-1} + |\xi'|^{-1}} \frac{d\lambda}{\lambda^2} \\ &\lesssim \frac{\sqrt{\xi |\xi'|}}{\xi + |\xi'|} \\ &\lesssim 1, \end{aligned}$$

and also

$$\begin{aligned} \int_{|\lambda - \lambda_0| > \delta} \frac{|a'(\lambda)|}{|\lambda - \lambda_0|} d\lambda &\lesssim (\langle \xi \rangle \langle \xi' \rangle)^{-1/2} \left(\int_{\lambda > \xi^{-1} + |\xi'|^{-1}} \frac{d\lambda}{\lambda^2 (\log |\lambda|)^2} \right. \\ &\quad \left. + \int_{\lambda > \xi^{-1} + |\xi'|^{-1}} \frac{d\lambda}{\lambda^3} (\xi^{-1} + |\xi'|^{-1}) \right) \\ &\lesssim \frac{\sqrt{\xi |\xi'|}}{\xi + |\xi'|} \\ &\lesssim 1. \end{aligned}$$

Case 3: $|\lambda_0| \gg 1$, $|\lambda_0| \gtrsim \xi^{-1} + |\xi'|^{-1}$.

In this case, $|\lambda - \lambda_0| \sim |\lambda_0| \gg 1$. Thus,

$$\int \frac{|a(\lambda)|}{|\lambda - \lambda_0|^2 + t^{-1}} d\lambda \lesssim (\langle \xi \rangle \langle \xi' \rangle)^{-1/2} \frac{1}{\lambda_0^2 + t^{-1}} \lesssim 1$$

as well as, see (2.54),

$$\begin{aligned} \int_{|\lambda - \lambda_0| > \delta} \frac{|a'(\lambda)|}{|\lambda - \lambda_0|} d\lambda &\lesssim \int \frac{\chi(\lambda)}{|\lambda|(\log |\lambda|)^2} d\lambda \frac{(\langle \xi \rangle \langle \xi' \rangle)^{-1/2}}{\lambda_0} \\ &\quad + \int \frac{1}{\lambda^2} \chi_{[|\lambda| > \xi^{-1} + |\xi'|^{-1}]} \frac{d\lambda}{\lambda_0} \frac{\xi + |\xi'|}{(\xi |\xi'|)^{3/2}} \\ &\lesssim 1, \end{aligned}$$

and the lemma is proved. \square

Now we turn to the estimate of the oscillatory integral for the case $\xi\lambda > 1$ and $|\xi'\lambda| < 1$.

Lemma 2.19. *For all $t > 0$*

$$(2.55) \quad \sup_{\xi > 0 > \xi'} \left| (\langle \xi \rangle \langle \xi' \rangle)^{-1/2} \int_{-\infty}^{\infty} e^{it\lambda^2} \frac{\lambda \chi(\lambda)}{W(\lambda)} \chi_{[\xi\lambda > 1, |\xi'\lambda| < 1]} f_+(\xi, \lambda) f_-(\xi', \lambda) d\lambda \right| \lesssim t^{-1}$$

and similarly with $\chi_{[|\xi\lambda| < 1, \xi'\lambda < -1]}$.

Proof. As before, we write $f_+(\xi, \lambda) = e^{i\xi\lambda} m_+(\xi, \lambda)$. But because of $|\xi'\lambda| < 1$ we use the representation

$$f_-(\xi', \lambda) = a_-(\lambda) u_0(\xi, \lambda) + b_-(\lambda) u_1(\xi, \lambda).$$

In particular,

$$|f_-(\xi', \lambda)| \lesssim \sqrt{|\lambda| \langle \xi' \rangle} |\log |\lambda||.$$

Moreover, from (2.35) and (2.44),

$$|\partial_\lambda f_-(\xi', \lambda)| \lesssim \langle \xi' \rangle^{1/2} |\lambda|^{-1/2} |\log |\lambda||$$

provided $|\xi'\lambda| < 1$.

We apply (2.50) with $\phi(\lambda) = \lambda^2 + \frac{\xi}{t}\lambda$ and

$$a(\lambda) = \frac{\lambda \chi(\lambda)}{W(\lambda)} (\langle \xi \rangle \langle \xi' \rangle)^{-1/2} \chi_{[\xi\lambda > 1, |\xi'\lambda| < 1]} m_+(\xi, \lambda) f_-(\xi', \lambda).$$

By the preceding,

$$(2.56) \quad |a(\lambda)| \lesssim \frac{|\lambda|^{1/2}}{\sqrt{\langle \xi \rangle}} \chi(\lambda) \chi_{[\xi\lambda > 1, |\xi'\lambda| < 1]}$$

and

$$(2.57) \quad |a'(\lambda)| \lesssim (|\lambda| \langle \xi \rangle)^{-1/2} \chi(\lambda) \chi_{[\xi\lambda > 1, |\xi'\lambda| < 1]}.$$

Case 1: $|\lambda_0| \lesssim 1$, $|\xi\lambda_0| \gtrsim 1$.

Note in particular $|\xi| \gtrsim 1$. Here $\lambda_0 = -\frac{\xi}{2t}$. By (2.56),

$$\begin{aligned} \int \frac{|a(\lambda)|}{|\lambda - \lambda_0|^2 + t^{-1}} d\lambda &\lesssim \langle \xi \rangle^{-1/2} \int \frac{\sqrt{|\lambda|}}{|\lambda - \lambda_0|^2 + t^{-1}} d\lambda \\ &\lesssim \langle \xi \rangle^{-1/2} |\lambda_0|^{1/2} \int \frac{d\lambda}{|\lambda - \lambda_0|^2 + t^{-1}} + \langle \xi \rangle^{-1/2} \int \frac{|\lambda|^{1/2}}{|\lambda|^2 + t^{-1}} d\lambda \\ &\lesssim \langle \xi \rangle^{-1/2} t^{1/2} \left(\frac{\xi}{t} \right)^{1/2} + \langle \xi \rangle^{-1/2} t^{1/4} \\ &\lesssim 1 \end{aligned}$$

Here we used that $|\xi \lambda_0| = \frac{\xi^2}{2t} \gtrsim 1$.

Next, write via (2.57)

$$(2.58) \quad \int_{|\lambda - \lambda_0| > \delta} \frac{|a'(\lambda)|}{|\lambda - \lambda_0|} d\lambda \lesssim \langle \xi \rangle^{-\frac{1}{2}} \int_{|\lambda - \lambda_0| > \delta} \frac{1}{|\lambda|^{\frac{1}{2}} |\lambda - \lambda_0|} \chi_{[\xi \lambda > 1, |\xi' \lambda| < 1]} d\lambda.$$

Distinguish the cases $\frac{1}{10}|\lambda| > |\lambda - \lambda_0|$ and $\frac{1}{10}|\lambda| \leq |\lambda - \lambda_0|$ in the integral on the right-hand side. This yields

$$\begin{aligned} (2.58) &\lesssim \langle \xi \rangle^{-1/2} \int_{|\lambda - \lambda_0| > \delta} \frac{d\lambda}{|\lambda - \lambda_0|^{3/2}} + \langle \xi \rangle^{-1/2} \int_{|\lambda| \lesssim |\lambda_0|} \frac{d\lambda}{|\lambda|^{1/2}} |\lambda_0|^{-1} \\ &\quad + \langle \xi \rangle^{-1/2} \int_{|\lambda| > |\lambda_0|} \frac{d\lambda}{|\lambda|^{3/2}} \\ &\lesssim \langle \xi \rangle^{-1/2} \delta^{-1/2} + \langle \xi \rangle^{-1/2} |\lambda_0|^{-1/2} \\ &\lesssim \left(\frac{t}{\xi^2} \right)^{1/4} + |\xi \lambda_0|^{-1/2} \\ &\lesssim 1. \end{aligned}$$

Case 2: $|\lambda_0| \lesssim 1$, $|\xi \lambda_0| \ll 1$.

In that case, $|\lambda - \lambda_0| \sim |\lambda|$ on the support of a . Consequently,

$$\int \frac{|a(\lambda)|}{|\lambda - \lambda_0|^2 + t^{-1}} d\lambda \lesssim \langle \xi \rangle^{-\frac{1}{2}} \int_{|\xi|^{-1}}^{\infty} |\lambda|^{-\frac{3}{2}} d\lambda \lesssim 1.$$

Moreover,

$$\int_{|\lambda - \lambda_0| > \delta} \frac{|a'(\lambda)|}{|\lambda - \lambda_0|} d\lambda \lesssim \int_{|\xi|^{-1}}^{\infty} \frac{(|\lambda| \langle \xi \rangle)^{-\frac{1}{2}}}{|\lambda|} d\lambda \lesssim 1.$$

Case 3: $|\lambda_0| \gg 1$.

In that case, $|\lambda - \lambda_0| \sim |\lambda_0|$ on $\text{supp}(a)$. Since $|a(\lambda)| \lesssim 1$ by (2.56), it follows that

$$\int \frac{|a(\lambda)|}{|\lambda - \lambda_0|^2 + t^{-1}} d\lambda \lesssim 1.$$

Similarly, since $|a'(\lambda)| \lesssim (\xi|\lambda|)^{-\frac{1}{2}}$, it follows that

$$\int_{|\lambda - \lambda_0| > \delta} \frac{|a'(\lambda)|}{|\lambda - \lambda_0|} d\lambda \lesssim \int \frac{(|\lambda| \langle \xi \rangle)^{-\frac{1}{2}}}{|\lambda_0|} \chi(\lambda) d\lambda \lesssim 1.$$

This proves (2.55). The other case $\chi_{[|\xi\lambda| < 1, \xi'\lambda < -1]}$ is treated in an analogous fashion. \square

The remaining cases for the small energy part of (2.12) are $\xi > \xi' > |\lambda|^{-1}$ and $\xi' < \xi < -|\lambda|^{-1}$. By symmetry it will suffice to treat the former case. As usual, we need to consider reflection and transmission coefficients, therefore we write

$$(2.59) \quad f_-(\xi, \lambda) = \alpha_-(\lambda) f_+(\xi, \lambda) + \beta_-(\lambda) \overline{f_+(\xi, \lambda)}.$$

Then, with $W(\lambda) = W(f_+(\cdot, \lambda), f_-(\cdot, \lambda))$,

$$W(\lambda) = \beta_-(\lambda) W(f_+(\cdot, \lambda), \overline{f_+(\cdot, \lambda)}) = -2i\lambda\beta_-(\lambda)$$

and

$$\begin{aligned} W(f_-(\cdot, \lambda), \overline{f_+(\cdot, \lambda)}) &= \alpha_-(\lambda) W(f_+(\cdot, \lambda), \overline{f_+(\cdot, \lambda)}) \\ &= -2i\lambda\alpha_-(\lambda). \end{aligned}$$

Thus, when $\lambda > 0$ is small,

$$(2.60) \quad \beta_-(\lambda) = i \left(1 + ic_3 + i\frac{2}{\pi} \log \lambda \right) + O(|\lambda|^{\frac{1}{2}-\varepsilon})$$

and

$$\begin{aligned} \alpha_-(\lambda) &= \frac{1}{-2i\lambda} W(a_+(\lambda)u_0(\cdot, \lambda) - b_+(\lambda)u_1(\cdot, \lambda), \overline{a_+(\lambda)u_0(\cdot, \lambda) + b_+(\lambda)u_1(\cdot, \lambda)}) \\ &= \frac{1}{-2i\lambda} (a_+ \overline{b_+}(\lambda) + \overline{a_+}(\lambda)b_+(\lambda)) \\ &= \frac{i}{\lambda} \text{Re}(a_+ \overline{b_+}(\lambda)) \\ &= \frac{i}{\lambda} \text{Re} \left(-i|c_0|^2 c_1 \lambda (1 + ic_1 \log \lambda + ic_3) + O(\lambda^{\frac{3}{2}-\varepsilon}) \right) \\ (2.61) \quad &= i \left(\frac{2}{\pi} \log \lambda + c_3 \right) + O(\lambda^{\frac{1}{2}-\varepsilon}). \end{aligned}$$

In passing, we remark that $1 + |\alpha_-|^2 = |\beta_-|^2$. Finally, it follows from Corollary 2.13 that the O -terms can be differentiated once in λ ; they then become $O(\lambda^{-\frac{1}{2}-\varepsilon})$, $\varepsilon > 0$ arbitrary.

Lemma 2.20. *For any $t > 0$*

$$(2.62) \quad \sup_{\xi > \xi' > 0} \left| \langle \langle \xi \rangle \langle \xi' \rangle \rangle^{-\frac{1}{2}} \int e^{it\lambda^2} \frac{\lambda \chi(\lambda)}{W(\lambda)} \chi_{[|\xi'\lambda| > 1]} f_+(\xi, \lambda) f_-(\xi', \lambda) d\lambda \right| \lesssim t^{-1}$$

and similarly for $\sup_{\xi' < \xi < 0}$ and $\chi_{[|\xi\lambda| > 1]}$.

Proof. Using (2.59), we reduce (2.62) to two estimates:

$$(2.63) \quad \sup_{\xi > \xi' > 0} \left| (\langle \xi \rangle \langle \xi' \rangle)^{-\frac{1}{2}} \int e^{it\lambda^2} \frac{\lambda \chi(\lambda)}{W(\lambda)} \chi_{[\xi'|\lambda|>1]} e^{i\lambda(\xi+\xi')} m_+(\xi, \lambda) m_+(\xi', \lambda) \alpha_-(\lambda) d\lambda \right| \lesssim t^{-1}$$

and

$$(2.64) \quad \sup_{\xi > \xi' > 0} \left| (\langle \xi \rangle \langle \xi' \rangle)^{-\frac{1}{2}} \int e^{it\lambda^2} e^{i\lambda(\xi-\xi')} \frac{\lambda \chi(\lambda)}{W(\lambda)} \beta_-(\lambda) \chi_{[\xi'|\lambda|>1]} m_+(\xi, \lambda) \overline{m_+(\xi', \lambda)} d\lambda \right| \lesssim t^{-1}$$

We apply (2.50) to (2.63) with fixed $\xi > \xi' > 0$ and

$$\begin{aligned} \phi(\lambda) &= \lambda^2 + \frac{\lambda}{t}(\xi + \xi'), \\ a(\lambda) &= (\langle \xi \rangle \langle \xi' \rangle)^{-\frac{1}{2}} \frac{\lambda \chi(\lambda)}{W(\lambda)} \chi_{[\xi'|\lambda|>1]} \alpha_-(\lambda) m_+(\xi, \lambda) m_+(\xi', \lambda). \end{aligned}$$

Then from (2.61),

$$(2.65) \quad |a(\lambda)| \lesssim (\langle \xi \rangle \langle \xi' \rangle)^{-\frac{1}{2}} \chi(\lambda) \chi_{[\xi'|\lambda|>1]}$$

and from our derivative bounds on W , α_- , and $m_+(\xi, \lambda)$, see (2.46) for the latter, we conclude that

$$(2.66) \quad |a'(\lambda)| \lesssim |\lambda|^{-1} (\langle \xi \rangle \langle \xi' \rangle)^{-\frac{1}{2}} \chi(\lambda) \chi_{[\xi'|\lambda|>1]}.$$

Case 1: Suppose $|\lambda_0| \lesssim 1$ and $|\xi' \lambda_0| > 1$, where $\lambda_0 = -\frac{\xi+\xi'}{2t}$. Note $\xi > \xi' \gtrsim 1$. Then

$$\begin{aligned} \int \frac{|a(\lambda)|}{|\lambda - \lambda_0|^2 + t^{-1}} d\lambda &\lesssim (\langle \xi \rangle \langle \xi' \rangle)^{-\frac{1}{2}} \int \frac{d\lambda}{|\lambda - \lambda_0|^2 + t^{-1}} \\ &\lesssim \sqrt{\frac{t}{\xi \xi'}} \lesssim 1 \end{aligned}$$

since $|\xi' \lambda_0| \sim \frac{\xi \xi'}{t} > 1$.

As for the derivative term in (2.50), we infer from (2.66) that

$$(2.67) \quad \int_{|\lambda - \lambda_0| > \delta} \frac{|a'(\lambda)|}{|\lambda - \lambda_0|} d\lambda \lesssim (\langle \xi \rangle \langle \xi' \rangle)^{-\frac{1}{2}} \int_{|\lambda - \lambda_0| > \delta} \frac{d\lambda}{|\lambda| |\lambda - \lambda_0|} \chi_{[|\lambda \xi'| > 1]}$$

Again, we need to distinguish between $|\lambda - \lambda_0| > \frac{1}{10}|\lambda_0|$ and $|\lambda - \lambda_0| < \frac{1}{10}|\lambda_0|$. Thus, since $\xi\xi' > t$,

$$\begin{aligned} (2.67) &\lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-\frac{1}{2}} \int_{1/\xi'}^{\infty} \frac{d\lambda}{\lambda^2} + (\langle\xi\rangle\langle\xi'\rangle)^{-1/2} |\lambda_0|^{-1} \log\left(t^{1/2}|\lambda_0|\right) \\ &\lesssim 1 + \frac{t^{\frac{1}{2}}}{\xi} \log\left(\frac{\xi}{t^{1/2}}\right) \\ &\lesssim 1 \end{aligned}$$

since also $\xi^2 > t$.

Case 2: $|\lambda_0| \lesssim 1$, $|\lambda_0| \ll \frac{1}{\xi'}$.

Then $|\lambda - \lambda_0| \sim |\lambda|$ on the support of $a(\lambda)$. Hence,

$$\int \frac{|a(\lambda)|}{|\lambda - \lambda_0|^2 + t^{-1}} d\lambda \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-\frac{1}{2}} \int_{1/\xi'}^{\infty} \frac{d\lambda}{\lambda^2} \lesssim \sqrt{\frac{\xi'}{\langle\xi\rangle}} < 1$$

and

$$\int_{|\lambda - \lambda_0| > \delta} \frac{|a'(\lambda)|}{|\lambda - \lambda_0|} d\lambda \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-\frac{1}{2}} \int_{1/\xi'}^{\infty} \frac{d\lambda}{\lambda^2} < 1.$$

Case 3: $|\lambda_0| \gg 1$, $|\lambda_0| \gtrsim \frac{1}{\xi'}$.

Then $|\lambda - \lambda_0| \sim |\lambda_0|$ on $\text{supp}(a)$. Therefore, $|a(\lambda)| \lesssim 1$ implies that

$$\int \frac{|a(\lambda)|}{|\lambda - \lambda_0|^2 + t^{-1}} d\lambda \lesssim 1$$

and

$$\begin{aligned} \int_{|\lambda - \lambda_0| > \delta} \frac{|a'(\lambda)|}{|\lambda - \lambda_0|} d\lambda &\lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-\frac{1}{2}} |\lambda_0|^{-1} \int_{\frac{1}{\langle\xi'\rangle}}^1 \frac{d\lambda}{|\lambda|} \\ &\lesssim \frac{1}{|\lambda_0|} \frac{1}{\langle\xi'\rangle} \log\langle\xi'\rangle \\ &\lesssim 1. \end{aligned}$$

This concludes the proof of (2.63). (2.64) is completely analogous, as is the case of $\xi' < \xi < 0$, $|\xi\lambda| > 1$. \square

We are done with the contributions of small λ in (2.12). To conclude the proof of Theorem 1.2 it suffices to prove the following statement.

Lemma 2.21. *For all $t > 0$,*

$$(2.68) \quad \sup_{\xi > \xi'} \left| (\langle\xi\rangle\langle\xi'\rangle)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{it\lambda^2} \frac{\lambda(1 - \chi)(\lambda)}{W(\lambda)} f_+(\xi, \lambda) f_-(\xi', \lambda) d\lambda \right| \lesssim t^{-1}.$$

Proof. We observed above, see (2.59), that $W(\lambda) = -2i\lambda\beta_-(\lambda)$. Since $|\beta_-(\lambda)| \geq 1$, this implies that $|W(\lambda)| \geq 2|\lambda|$. In particular, $W(\lambda) \neq 0$ for every $\lambda \neq 0$.

In order to prove (2.68), we will need to distinguish the cases $\xi > 0 > \xi'$, $\xi > \xi' > 0$, $0 > \xi > \xi'$. By symmetry, it will suffice to consider the first two.

Case 1: $\xi > 0 > \xi'$.

In this case we need to prove that

$$(2.69) \quad \sup_{\xi > 0 > \xi'} \left| (\langle \xi \rangle \langle \xi' \rangle)^{-\frac{1}{2}} \int e^{it[\lambda^2 + \frac{\xi - \xi'}{t}\lambda]} \frac{\lambda(1 - \chi)(\lambda)}{W(\lambda)} m_+(\xi, \lambda) m_-(\xi', \lambda) d\lambda \right| \lesssim t^{-1}.$$

Apply (2.50) with $\phi(\lambda) = \lambda^2 + \frac{\xi - \xi'}{t}\lambda$ and

$$a(\lambda) = (\langle \xi \rangle \langle \xi' \rangle)^{-\frac{1}{2}} \frac{\lambda(1 - \chi)(\lambda)}{W(\lambda)} m_+(\xi, \lambda) m_-(\xi', \lambda).$$

Hence, with $\lambda_0 = -\frac{\xi - \xi'}{2t}$,

$$(2.69) \lesssim t^{-1} \left(\int \frac{|a(\lambda)|}{|\lambda - \lambda_0|^2 + t^{-1}} d\lambda + \int_{|\lambda - \lambda_0| > \delta} \frac{|a'(\lambda)|}{|\lambda - \lambda_0|} d\lambda \right) \\ =: t^{-1}(A + B).$$

If $|\lambda_0| \ll 1$, then

$$A \lesssim \|a\|_\infty \lesssim 1.$$

On the other hand, if $|\lambda_0| \gtrsim 1$, then $\xi + |\xi'| \gtrsim t$ so that

$$A \lesssim t^{\frac{1}{2}} \|a\|_\infty \lesssim t^{\frac{1}{2}} (\langle \xi \rangle \langle \xi' \rangle)^{-\frac{1}{2}} \lesssim \sqrt{\frac{t}{\langle \xi \rangle \langle \xi' \rangle}} \lesssim 1.$$

Here we used that

$$\sup_{\xi} \sup_{|\lambda| \gtrsim 1} |m_{\pm}(\xi, \lambda)| \lesssim 1$$

which follows from the fact that

$$(2.70) \quad m_+(\xi, \lambda) = 1 + \int_{\xi}^{\infty} \frac{1 - e^{-2i(\tilde{\xi} - \xi)\lambda}}{2i\lambda} V(\tilde{\xi}) m_+(\tilde{\xi}, \lambda) d\tilde{\xi}$$

with

$$V(\tilde{\xi}) = \left(\frac{1}{4}\omega^2 + \frac{1}{2}\dot{\omega} \right) (\tilde{\xi}) = O(\langle \tilde{\xi} \rangle^{-2}).$$

Moreover, from our assumptions on $r(x)$ we recall that

$$\left| \frac{d^l}{d\xi^l} V(\xi) \right| \lesssim \langle \xi \rangle^{-2-l}, \quad l \geq 0.$$

We shall need these bounds to estimate B above. From (2.70), for $\xi \geq 0$

$$m_+(\xi, \lambda) = 1 + O(\lambda^{-1} \langle \xi \rangle^{-1})$$

as well as for $\xi \geq 0$

$$(2.71) \quad \partial_\xi^j m_+(\xi, \lambda) = O(\lambda^{-1} \langle \xi \rangle^{-1}), \quad j = 1, 2$$

$$(2.72) \quad \partial_\lambda m_+(\xi, \lambda) = O(\lambda^{-2} \langle \xi \rangle^{-1})$$

$$(2.73) \quad \partial_\lambda \partial_\xi m_+(\xi, \lambda) = O(\lambda^{-2} \langle \xi \rangle^{-1})$$

To verify (2.71), one checks that

$$(2.74) \quad \begin{aligned} \partial_\xi m_+(\xi, \lambda) &= \frac{1}{2i\lambda} \int_\xi^\infty e^{2i(\xi-\tilde{\xi})\lambda} \partial_{\tilde{\xi}} \left[(\xi - \tilde{\xi}) V(\tilde{\xi}) \right] m_+(\tilde{\xi}, \lambda) d\tilde{\xi} \\ &\quad + \frac{1}{2i\lambda} \int_\xi^\infty e^{2i(\xi-\tilde{\xi})\lambda} (\xi - \tilde{\xi}) V(\tilde{\xi}) \partial_{\tilde{\xi}} m_+(\tilde{\xi}, \lambda) d\tilde{\xi}. \end{aligned}$$

By our estimates on V , the integral in (2.74) is $O(\lambda^{-1} \langle \xi \rangle^{-1})$ and (2.71) follows. For (2.72) we compute

$$\begin{aligned} \partial_\lambda m_+(\xi, \lambda) &= - \int_\xi^\infty \frac{1 - e^{2i(\xi-\tilde{\xi})\lambda}}{2i\lambda^2} V(\tilde{\xi}) m_+(\tilde{\xi}, \lambda) d\tilde{\xi} \\ &\quad + \frac{1}{2i\lambda^2} \int_\xi^\infty e^{2i(\xi-\tilde{\xi})\lambda} \partial_{\tilde{\xi}} \left[(\xi - \tilde{\xi}) V(\tilde{\xi}) m_+(\tilde{\xi}, \lambda) \right] d\tilde{\xi} \\ &\quad + \int_\xi^\infty \frac{1 - e^{2i(\xi-\tilde{\xi})\lambda}}{2i\lambda} V(\tilde{\xi}) \partial_\lambda m_+(\tilde{\xi}, \lambda) d\tilde{\xi} \end{aligned}$$

so that

$$\partial_\lambda m_+(\xi, \lambda) = O(\lambda^{-2} \langle \xi \rangle^{-1})$$

as claimed.

Finally, compute

$$\begin{aligned} \partial_{\xi\lambda}^2 m_+(\xi, \lambda) &= \frac{1}{\lambda} \int_\xi^\infty e^{2i(\xi-\tilde{\xi})\lambda} V(\tilde{\xi}) m_+(\tilde{\xi}, \lambda) d\tilde{\xi} \\ &\quad + \frac{1}{2i\lambda^2} V(\xi) m_+(\xi, \lambda) + \frac{1}{2i\lambda} \int_\xi^\infty e^{2i(\xi-\tilde{\xi})\lambda} \partial_{\tilde{\xi}} [(\xi - \tilde{\xi}) V m_+] d\tilde{\xi} \\ &\quad + \frac{1}{2i\lambda^2} \int_\xi^\infty e^{2i(\xi-\tilde{\xi})\lambda} \partial_{\tilde{\xi}} [V(\tilde{\xi}) m_+(\tilde{\xi}, \lambda)] d\tilde{\xi} \\ &\quad - \int_\xi^\infty e^{2i(\xi-\tilde{\xi})\lambda} V(\tilde{\xi}) \partial_\lambda m_+(\tilde{\xi}, \lambda) d\tilde{\xi} \end{aligned}$$

Integrating by parts in the first, third, and last terms yields the desired estimate. As a corollary, we obtain (take $\xi = 0$)

$$\begin{aligned} W(\lambda) &= W(f_+(\cdot, \lambda), f_-(\cdot, \lambda)) \\ &= m_+(\xi, \lambda)[m'_-(\xi, \lambda) - i\lambda m_-(\xi, \lambda)] - m_-(\xi, \lambda)[m'_+(\xi, \lambda) + i\lambda m_+(\xi, \lambda)] \\ &= -2i\lambda(1 + O(\lambda^{-1})) + O(\lambda^{-1}) \\ &= -2i\lambda + O(1) \end{aligned}$$

and

$$W'(\lambda) = -2i + O(\lambda^{-1})$$

as $|\lambda| \rightarrow \infty$.

Next, we estimate B . First, we conclude from our bounds on $W(\lambda)$ and $m_+(\xi, \lambda)$ as well as $m_-(\xi', \lambda)$ that

$$|a'(\lambda)| \lesssim (\langle \xi \rangle \langle \xi' \rangle)^{-\frac{1}{2}} \chi_{[|\lambda| \gtrsim 1]} |\lambda|^{-1}.$$

Let us first consider the case where $|\lambda_0| \gtrsim 1$. Then

$$\begin{aligned} B &\lesssim (\langle \xi \rangle \langle \xi' \rangle)^{-1/2} \int_{\substack{(|\lambda - \lambda_0| > \delta) \\ |\lambda| \gtrsim 1}} \frac{d\lambda}{|\lambda| |\lambda - \lambda_0|} \\ &\lesssim (\langle \xi \rangle \langle \xi' \rangle)^{-1/2} \left\{ \int_1^\infty \frac{d\lambda}{\lambda^2} + \frac{1}{|\lambda_0|} \int_{\frac{|\lambda_0|}{5} > |\lambda - \lambda_0| > \delta} \frac{d\lambda}{|\lambda - \lambda_0|} \right\} \\ &\lesssim 1 + \sqrt{\frac{t}{\langle \xi \rangle \langle \xi' \rangle}} \frac{1}{|\lambda_0| t^{1/2}} \log_+(\lambda_0 t^{1/2}) \\ &\lesssim 1 \end{aligned}$$

Here we used that $\frac{t}{\langle \xi \rangle \langle \xi' \rangle} \lesssim 1$ which follows from $|\lambda_0| \gtrsim 1$.

If $|\lambda_0| \ll 1$, then $|\lambda - \lambda_0| \sim |\lambda|$ on the support of a ; thus $B \lesssim 1$ trivially. This finishes the case $\xi > 0 > \xi'$.

Case 2: To deal with the case $\xi > \xi' > 0$, we use (2.59). Thus,

$$f_-(\xi', \lambda) = \alpha_-(\lambda) f_+(\xi', \lambda) + \beta_-(\lambda) \overline{f_+(\xi', \lambda)}$$

where

$$\begin{aligned} \alpha_-(\lambda) &= \frac{W(f_-(\cdot, \lambda), \overline{f_+(\cdot, \lambda)})}{-2i\lambda} \\ \beta_-(\lambda) &= \frac{W(f_+(\cdot, \lambda), f_-(\cdot, \lambda))}{-2i\lambda} = \frac{W(\lambda)}{-2i\lambda} \end{aligned}$$

From our large λ asymptotics of $W(\lambda)$ we deduce that $\beta_-(\lambda) = 1 + O(\lambda^{-1})$ and $\beta'_-(\lambda) = O(\lambda^{-2})$. For $\alpha_-(\lambda)$ we calculate, again at $\xi = 0$,

$$\begin{aligned} W(f_-(\cdot, \lambda), \overline{f_+(\cdot, \lambda)}) &= m_-(\xi, \lambda)(\overline{m'_+(\xi, \lambda)} - 2i\lambda\overline{m_+(\xi, \lambda)}) \\ &\quad - \overline{m_+(\xi, \lambda)}(m'_-(\xi, \lambda) - 2i\lambda m_-(\xi, \lambda)) \\ &= m_-(\xi, \lambda)\overline{m'_+(\xi, \lambda)} - m'_-(\xi, \lambda)\overline{m_+(\xi, \lambda)} \\ &= O(\lambda^{-1}) \end{aligned}$$

so that

$$\alpha_-(\lambda) = O(\lambda^{-2})$$

with

$$\alpha'_-(\lambda) = O(\lambda^{-3}).$$

Thus, we are left with showing that

$$(2.75) \quad \sup_{\xi > \xi' > 0} \left| \langle \langle \xi \rangle \langle \xi' \rangle \rangle^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{it\lambda^2} e^{i\lambda(\xi + \xi')} \frac{\lambda(1 - \chi(\lambda))}{W(\lambda)} \alpha_-(\lambda) m_+(\xi, \lambda) m_+(\xi', \lambda) d\lambda \right| \lesssim t^{-1}$$

and

$$(2.76) \quad \sup_{\xi > \xi' > 0} \left| \langle \langle \xi \rangle \langle \xi' \rangle \rangle^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{it\lambda^2} e^{i\lambda(\xi - \xi')} \frac{\lambda(1 - \chi(\lambda))}{W(\lambda)} \beta_-(\lambda) m_+(\xi, \lambda) m_+(\bar{\xi}', \lambda) d\lambda \right| \lesssim t^{-1}$$

for any $t > 0$.

This, however, follows by means of the exact same arguments which we use to prove (2.69). Note that in (2.74) the critical point of the phase is

$$\lambda_0 = -\frac{\xi + \xi'}{2t}$$

whereas in (2.75) it is

$$\lambda_0 = -\frac{\xi - \xi'}{2t}.$$

In either case it follows from $|\lambda_0| \gtrsim 1$ that $\xi \gtrsim t$. Hence we can indeed argue as in case 1.

This finishes the proof of the lemma, and thus also Theorem 1.2. \square

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